13 → 15 ~~10~~ over

written 2022-04-04

cs473 Algorithms: Lecture 20 (2022-04-05)

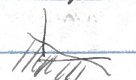
logistics:
- pset 8 due F17
- exam2 04-11 ← piazza

last lecture: linear programming - duality
- example - by hand / - mechanized
- def
- weak duality - canonical / - non-canonical
- dual of max-flow = min-cut [strong duality ⇒ max-flow = min-cut]

today: linear programming

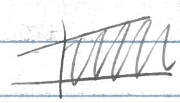
def: [II terms II] A *halfspace* is a set $\{x : \langle a, x\rangle \le b\}$ ($a \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}$)

eg $x + 2y \le 10$



A *polyhedron* is the intersection of a finite # of halfspaces $\{x : Ax \le b\}$ ($A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$)

eg



← ~~not~~ bounded

A *polytope* is a bounded polyhedron, ie $P \subseteq [-B, B]^n$ some $B > 0$. why? • P

$\pi_{\le k} : \mathbb{R}^n \to \mathbb{R}^k$ is the *projection* onto the first k coordinates $(x_1, \dots, x_k, \dots x_n) \mapsto (x_1, \dots x_k)$

thm [strong duality]: [II last lecture II] I [II primal II] $= \max \langle c, x\rangle$ st $Ax \le b$, $x \ge 0$, II [II dual II] $= \min \langle b, y\rangle$ st $A^T y \ge c$, $y \ge 0$

if I feasible, bounded ⇒ II is also, $|\text{I}| = |\text{II}|$

pf **idea**: define $P = \{(z, x) :$ [II z scalar II] $z - \langle c, x\rangle \le 0$, $Ax \le b$, $x \ge 0\}$ polyhedron

$P_z = \pi_z(P)$
$= \{z : z \le |\text{I}|\}$ $|\text{I}|$ well-defined!

IF P_z polyhedron defined by equations arising by $\boxed{\text{non-negative}}$ linear combinations of equations in P

⇒ "$z \le |\text{I}|$" is expressible by dual

⇒ dual provides $\boxed{\text{optimal}}$ ub on primal ⇒ strong duality ☐

Q: projections of polyhedra?

recall: [II linear alg II] gaussian elimination on linear equalities

eg [II linear equations II]
$$\begin{aligned} x + y &= 1 \\ + \; 2x - y &= 5 \\ \hline 3x \quad &= 6 \Rightarrow x = 2 \end{aligned}$$
valid equations on projection to x coord

idea: gaussian elimination on linear $\boxed{\text{in}}$ equalities
$$\begin{aligned} \alpha(\; x + y &\le 1) \\ + \; \beta(\; 2x - y &\le 5) \\ \hline (\alpha + 2\beta)x + (\alpha - \beta)y &\le \alpha + 5\beta \end{aligned}$$
$\alpha, \beta \ge 0$ ← must preserve sign of inequality

$(\alpha + 2\beta)$ only $= 0$ if $\alpha = \beta = 0$ ⇒ $\boxed{\text{cannot}}$ eliminate x

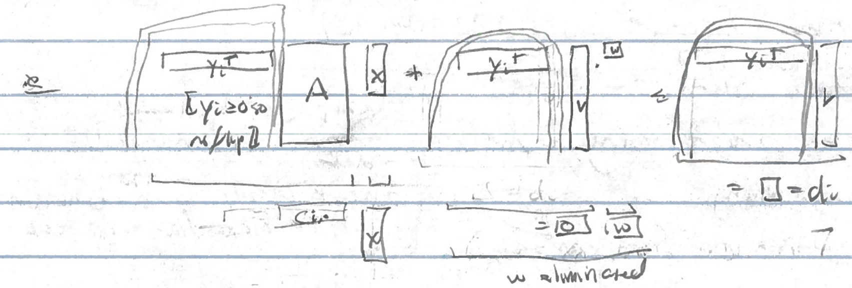
$\alpha = \beta = 1$ ⇒ $3x \le 6$ ⇒ $x \le 2$ eliminated y; valid inequality on projection to x-coord

thm [Fourier-Motzkin elimination]: [II generalize II] [II elim ~~one~~ var II] $P = \{x, w : Ax + vw \le b\}$, $A \in \mathbb{R}^{m \times n}$, [II $w \in \mathbb{R}$ II] $v \in \mathbb{R}^{m \times 1}$, $b \in \mathbb{R}^m$ polyhedron

then
- $\pi_x(P)$ [II proj onto first n coord II] is a polyhedron $Q = \{x : Cx \le d\}$ [II proj polyhedra is polyhedra II]
- "$Cx \le d$" is $\le m^2$ constraints on n variables [II $A \in \mathbb{R}^{m \times n}$ II] [II complexity of proj II]
- ↳ are $\boxed{\text{non-negative}}$ linear combinations of "$Ax + vw \le b$" [II above example II]

ie $C_{i,\cdot} = y_i^T A$ [II i-th equation II], $d_i = y_i^T b$, some $y_i \in \mathbb{R}^m_{\ge 0}$, w/ $y_i^T v = 0$ [II elim w II]

~~draw picture~~



pf: $P = \{Ax + vw \leq b\}$ has m equations, partition $[m] = S_+ \cup S_0 \cup S_-$

$S_+ = \{i : v_i > 0\}$ coeff in w [var solution] in ith equation

$S_0 = \{ \text{———} = 0\}$

$S_- = \{ \text{———} < 0\}$

construct "$Cx \leq d$" as $\{A_{i,\cdot}\, x \leq b_i\}_{i \in S_0}$ [did not involve w anyhow]

$$\cup \Big\{ \underbrace{\underset{\substack{\vee\\0}}{-v_i}\,(\overset{[j\text{th eqn}]}{A_{j,\cdot}\, x + v_j w \leq b_j}) + \underset{\substack{\vee\\0}}{(-v_j)}\,(\overset{[i\text{th eqn}]}{A_{i,\cdot}\, x + v_i w \leq b_i})}_{} \Big\}_{\substack{i \in S_+\\ j \in S_-}}$$

$$\equiv (v_i A_{j,\cdot} - v_j A_{i,\cdot})\, x + \underbrace{(v_i v_j - v_j v_i)}_{=0} \cdot w \leq \overset{[\text{scalar}]}{v_i b_j - v_j b_i}$$

2019 typo

clm: new equations are non-neg linear combinations of orig eqns [as in duality] [clear]

- do not involve w [clear: S_0 type S_+, S_- combination]

clm: at most $|S_0| + |S_+| \cdot |S_-|$ eqns [clear]

$$\leq (|S_0| + |S_+| + |S_-|)^2 = m^2$$

2019 typo

clm: $\pi_x(P) \subseteq Q = \{x : Cx \leq d\}$ [easy direction]

pf: $(x,w) \in P \Rightarrow Ax + vw \leq b \Rightarrow y^T(Ax + vw \leq b)$ any $y \geq 0$ [derived eqn are valid]

2019 typo

$$\equiv \underbrace{(y^T A)}_{=C_{i,\cdot}} x + \underbrace{\langle y^T, v\rangle}_{=0} w \leq \underbrace{\langle y^T, b\rangle}_{d_i} \equiv Cx \leq d$$

[choose y as above] all i ☐

clm: $\pi_x(P) \supseteq Q$ [hard direction]

pf: given $x \in Q$, equiv $Cx \leq d$, want $\exists w$ st $(x, w) \in P$ [undo projection] $\equiv Ax + vw \leq b$

$$\equiv v \cdot w \leq b - Ax,$$

[examine eqns defining $\mathbb{R}$] $\equiv \mathbb{R} = \{w : v \cdot w \leq b - Ax\}$ is non-empty

one-dimensional polyhedron

subclaim: $i \in S_0 \equiv v_i = 0$

$\Rightarrow (v \cdot w \leq b - Ax)_i \equiv (0 \leq b_i - A_{i,\cdot}\, x) \equiv A_{i,\cdot}\, x \leq b_i$

equation we added to "$Cx \leq d$"

$\Rightarrow$ is satisfied by $x \in Q$

≥ 0

$\Rightarrow$ does not constrain w

subclaim. $\{w : v \cdot w \leq \underbrace{b - Ax}_{u}\} = \{w : v_i w \leq u_i\}_{i \in S_+}$ [$\Rightarrow v_i > 0$] $= \{w \leq \frac{u_i}{v_i}\}_i$ [$v_i > 0$]

[intersection] $\cap \{ \text{———} \}_{i \in S_0}$ [$v_i = 0$] $= \mathbb{R}$ [above]

$= \{w : \max_{i \in S_-} \frac{u_i}{v_i} \leq w \leq \min_{i \in S_+} \frac{u_i}{v_i}\}$ $\cap \{ \text{———} \}_{i \in S_-}$ [$v_i < 0$] $= \{w \geq \frac{u_i}{v_i}\}_i$ [$v_i < 0$]

draw proj

subclaim: ↑ is always nonempty [exists way to undo projection]

pf: equiv $\max_{j\in S_-} \frac{u_j}{v_j} \le \min_{i\in S_+} \frac{u_i}{v_i}$ [nonempty interval]

equiv $\forall i\in S_+, j\in S_-: \frac{u_j}{v_j} \le \frac{u_i}{v_i}$ mult by $\underset{>0}{v_i}\underset{\le 0}{v_j} < 0$ so sign flips

$\equiv v_i u_j \ge v_j u_i$

[def of u] $\equiv v_i(b_j - A_{j,\cdot}x) \ge v_j(b_i - A_{i,\cdot}x)$

$\equiv (v_i A_{j,\cdot} - v_j A_{i,\cdot})x \le v_i b_j - v_j b_i$

← $\boxed{\text{true}}$ as is equation we added by def to "$Cx\le d$"

hence: $x\in Q \Rightarrow Cx\le d$

$\Rightarrow \forall i\in S_+, j\in S_-$

$\Rightarrow \max_{j\in S_-}\frac{u_j}{v_j} \le \min_{i\in S_+}\frac{u_i}{v_i} \Rightarrow \exists w \in [\max -, \min -]$

$\Rightarrow \{w : v\cdot w \le b - Ax\}$ nonempty

$\Rightarrow (x,w)\in P \Rightarrow x\in \pi_x(P)$ ∎

[induct]

cor: $P=\{x : Ax\le b\}$ polyhedron, A $m\times n$. Then $\pi_{\le k}(P)$ [proj to first k coord]

- is polyhedra $\{x_{\le k} : Cx_{\le k}\le d\}$ [first k coord]
- ↑ defined by $\le m^{2^{n-k}}$ constraints [square each step]
- "$Cx_{\le k}\le d$" are non-negative linear combinations of "$Ax\le b$" [as before, combine eqns w/ non-neg combos & elim all but first k vars]

[back to strong duality]

cor: Π $\max \langle c,x\rangle$ st $Ax\le b$ $\boxed{\text{LP}}$ canonical [clearly contains canonical] [all in x]

define polyhedron P by $z - \langle c,x\rangle \le 0$, $Ax\le b$ [in x, z], $P_z = \pi_z(P)$ [proj onto z]

then: Π unbounded $\equiv$ P_z has no constraints } z smaller than some feasible value

Π infeasible $\equiv$ P_z defined by "$0\le -1$"

Π feasible $\equiv$ P_z defined by "$z\le |\Pi|$"

pf: FM elimination $\Rightarrow$ P_z defined by $\boxed{\text{non-negative}}$ linear comb of $\{z - \langle c,x\rangle \le 0, Ax\le b\}$ only involving z

$\Rightarrow \{\alpha_i z \le \beta_i\}_i$ w/ $\alpha_i \ge 0$

clm: $\exists i$ $\alpha_i = 0$ $\beta_i < 0$ $\equiv \underset{=0}{\alpha_i} z \le \beta_i < 0 \equiv 0\le -1 \equiv \Pi$ infeasible non-neg comb of

clm: - can discard eqns w/ $\alpha_i = 0$, $\beta_i \ge 0$ $\equiv 0\le\beta_i$ [vacuous]

- if no remaining equations $\Rightarrow P_z = \mathbb{R} \equiv \Pi$ unbounded
- else, $P_z = \{z : \forall i\ z \le \beta_i/\alpha_i\}$ $\alpha_i \ge 0$ [non-neg linear comb]

$= \{z : z \le \min_i \beta_i/\alpha_i\}$ [is feasible value $|\Pi|$]

$\Rightarrow \Pi$ feasible, bounded, $\boxed{|\Pi|} = $ ↑ [exhausts options] ∎

[$\boxed{\text{canonical}}$ problems have no [illegible]]

cor: linear programming has finite time algo

sketch: apply Fourier-Motzkin elimination to $z - \langle c,x\rangle \le 0$, $Ax\le b$ to elim x, compute P_z as above, read off soln □

rmk: $\boxed{\text{terrible}}$ running, $\ge m^{\Theta(2^n)}$ for - m constraints - n variables

[note this is clear, dual w/ canonical form]

co [strong duality] : $\mathbb{I}$ [primal] max $\langle c,x\rangle$ st $Ax \le b$, $x \ge 0$, $\amalg$ [dual] min $\langle b,y\rangle$ st $y^TA \ge c$, $y \ge 0$

if $\mathbb{I}$ feasible, bounded $\Rightarrow$ $\amalg$ feasible bounded $|\mathbb{I}| = |\amalg|$

pf : $\mathbb{I}$ equiv to max $\langle c,x\rangle$ st $A'x \le b'$ yields $\begin{bmatrix} A \\ -I \end{bmatrix} [x] \le \begin{bmatrix} b \\ 0 \end{bmatrix}$

$\mathbb{I}$ feasible bounded $\Rightarrow$ FM an

eqn $\alpha z \le \beta$ w/
- $\alpha > 0$
- $|\mathbb{I}| = \beta/\alpha$ [non-neg comb]
- $(\alpha z \le \beta) = \alpha(z - \langle c,x\rangle) + (y')^T(A'x \le b')$, $y' \ge 0$

$\Rightarrow (y')^T A' = \alpha c$ [x elim]

$(y')^T b' = \beta$

$\Rightarrow (y'')^T A' = c$

$(y'')^T b' = \beta/\alpha = |\mathbb{I}|$ for $y'' = y'/\alpha \ge 0$ $\quad \alpha > 0$

note $A' = \begin{bmatrix} A \\ -I \end{bmatrix}$ $b' = \begin{bmatrix} b \\ 0 \end{bmatrix}$ $y'' = \begin{bmatrix} y \\ w \end{bmatrix}$

$\Rightarrow c = (y'')^T A' = y^TA - w^TI = y^TA - \underbrace{w^T}_{\ge 0}$ $\Rightarrow y^TA \ge c$, $y \ge 0$

$|\mathbb{I}| = (y'')^T b' = y^Tb - w^T \cdot 0 = y^Tb$ $\Rightarrow \langle b,y\rangle = |\mathbb{I}|$

$\Rightarrow$ $\amalg$ has feasible soln w/ value $|\mathbb{I}|$

$\amalg$ bounded $\Rightarrow$ $\mathbb{I}$ feasible

$\Rightarrow |\amalg| = |\mathbb{I}|$ [weak duality] [dual LPs last lecture]

co [max flow min cut] : max flow = min cut, for $\boxed{\text{arbitrary}}$ capacities [FF only does $\boxed{\text{integral}}$]

today : linear programming - Fourier Motzkin elimination [like gaussian elim, but
- eliminate 1 var via $\boxed{\text{non-neg}}$ linear comb $\boxed{\text{lin}}$ equalities]
- $\leftarrow$ k vars
- projections of polyhedra are polyhedra
- proof of strong duality
- $\Rightarrow$ maxflow = min cut

next lecture : linear programming

logistics : pset 8 due F 12

exam 2 04-11 w/ pizza