CS 473 Algorithms: Lecture 12 (2022-03-01)

logistics: - psets on F17

last lecture: - flows
- reductions to max flow
- circulations w/ demands
- ______________, w/ lower bounds
- bipartite matching w/ forced matches

today: randomized algo

Q: what is the "most realistic" model of computation?
- can run on actual computers
- captures abilities of actual computers
  - ↳ rand() ?

↳ randomized algo can use rand(k)
- O(1) time
- returns $0 \le i < k$ w/p $\frac{1}{k}$

Q: who will win the next US presidential election?

A: ask everyone
- ~ 328 million ppl
- ~ 235 million eligible voters
- ~ 138 million actual voters

fact: a poll of 738 uniformly random voters will
- estimate actual vote up to error ≤ 5%
- with probability ≥ 95%

Q: why rand algo?

A: - simpler algo, but analysis can be more complicated
- faster algo, but nonzero failure probability

sometimes [only] efficient algo known is randomized

Q: how to model randomized algo?

models: deterministic: deterministic algo f

worst case input x

$x \mapsto f(x)$

complexity $T(n) = \max_{|x|=n} T(x)$

def: Ω finite / countably-infinite set.

$\Pr : \Omega \to [0,1]$. then $(\Omega, \Pr)$ is

a discrete probability space if $\sum_{\omega \in \Omega} \Pr[\omega] = 1$

For an event $E \subseteq \Omega$ $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$

A random variable is a function $X : \Omega \to \mathbb{R}$, has

expectation $\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$

how to analyze rand algo:

correctness: output always correct

complexity: expected runtime as function of input size

model: probabilistic

det algo f

input $x \leftarrow \mathcal{D}_n$

dist over size n inputs

$x \mapsto f(x)$

complexity: $T(n) = \mathbb{E}_{x \leftarrow \mathcal{D}_n} T(x)$

rmk: - $\mathcal{D}$ is often unknown in real life

$\Rightarrow$ theory brittle

randomized algo: randomized algo

random in algo

worst case input x

$x \mapsto f(x)$

complexity $T(n) = \max_{|x|=n} \mathbb{E}[T(x)]$

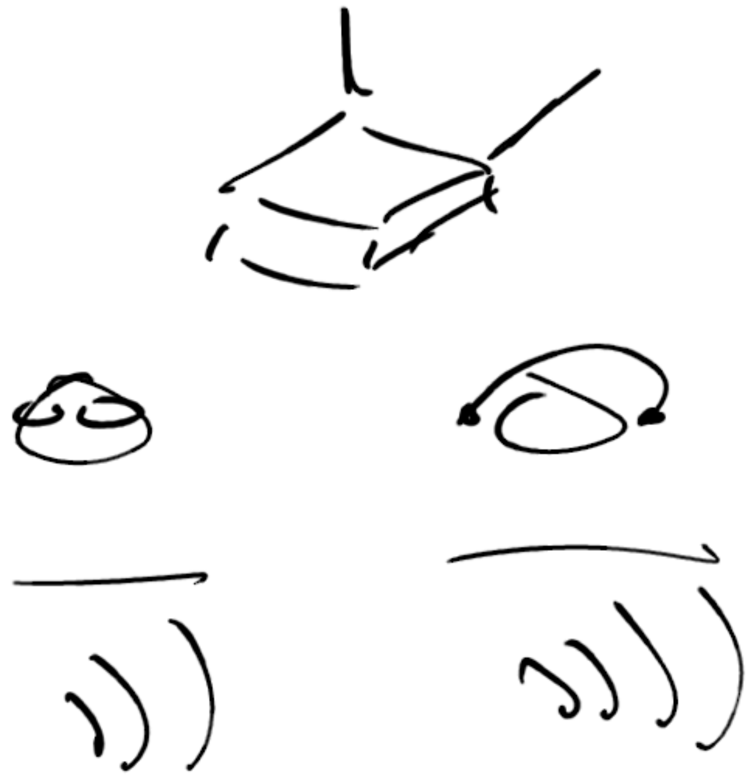
rmk: "right" notion

Q: how does wifi work?

⇒ collision!

Q: how to avoid collision?

idea: take turns



round 1

round 2 ⇒ collision!

idea: $\boxed{\text{assign}}$ turns

↳ requires $\boxed{\text{communication}}$ to assign!

Q: how to assign $\boxed{\text{without}}$ centralized setup?

i.e. break symmetry

idea: use $\boxed{\text{names}}$

Mitforbes → even length

⇒ talk in even rounds

alice → odd length

⇒ talk in odd rounds

no collisions!

$\boxed{\text{but}}$ needs more work

eg $|bob| = 3$ ⇒ collision

idea: use randomness!

def. the contention resolution process is, for

n people $P_1, \ldots, P_n$

in round j: each person tries to communicate

w/p p

(if P_i is $\boxed{\text{only}}$ person communicating, they succeed

else everyone fails

Q: how many rounds until everyone succeeds?

def: $X_{ij} = \begin{cases} 1 & P_i \text{ attempts communication in round } j \\ 0 & \text{else} \end{cases}$

$\{X_{ij}\}_{i,j}$ are $\boxed{\text{independent}}$

$\hookrightarrow \Pr[X_{ij} = \underset{\{0,1\}}{a} \wedge X_{i'j'} = \underset{\{0,1\}}{b}] = \Pr[X_{ij} = a] \cdot \Pr[X_{i'j'} = b]$

lem: $\mathbb{E}[X_{ij}] = 1 \cdot \underbrace{\Pr[X_{ij} = 1]}_{p} + 0 \cdot \underbrace{\Pr[X_{ij} = 0]}_{1-p}$

$= \Pr[X_{ij} = 1]$

$= p$

lem: $\mathbb{E}[\#\text{ people trying to communicate in round } j]$

$= \mathbb{E}\left[\sum_{i=1}^{n} X_{ij}\right] = \sum_{i=1}^{n} \mathbb{E}[X_{ij}]$

lem [linearity of expectation] any rand X, Y, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$= n \cdot p$

def: choose $p = 1/n$, so $= 1$

lem: $\{ P_i \text{ succeeds in round } j \}$

$$= \{ X_{ij} = 1 \} \cap \{ X_{i'j} = 0 \}_{i' \neq i}$$

cor: $\Pr[P_i \text{ succeeds round } j]$

$$= \Pr\left[X_{ij} = 1 \wedge \bigwedge_{i' \neq i} X_{i'j} = 0 \right]$$

$$\overset{\text{independence}}{=} \underbrace{\Pr[X_{ij} = 1]}_{= p = \frac{1}{n}} \cdot \prod_{i' \neq i} \underbrace{\Pr[X_{i'j} = 0]}_{\substack{1-p \\ = 1 - 1/n}}$$

$$= \underline{\tfrac{1}{n} \left(1 - \tfrac{1}{n} \right)^{n-1}}$$

idea: avoid $\boxed{\text{exact}}$ expressions

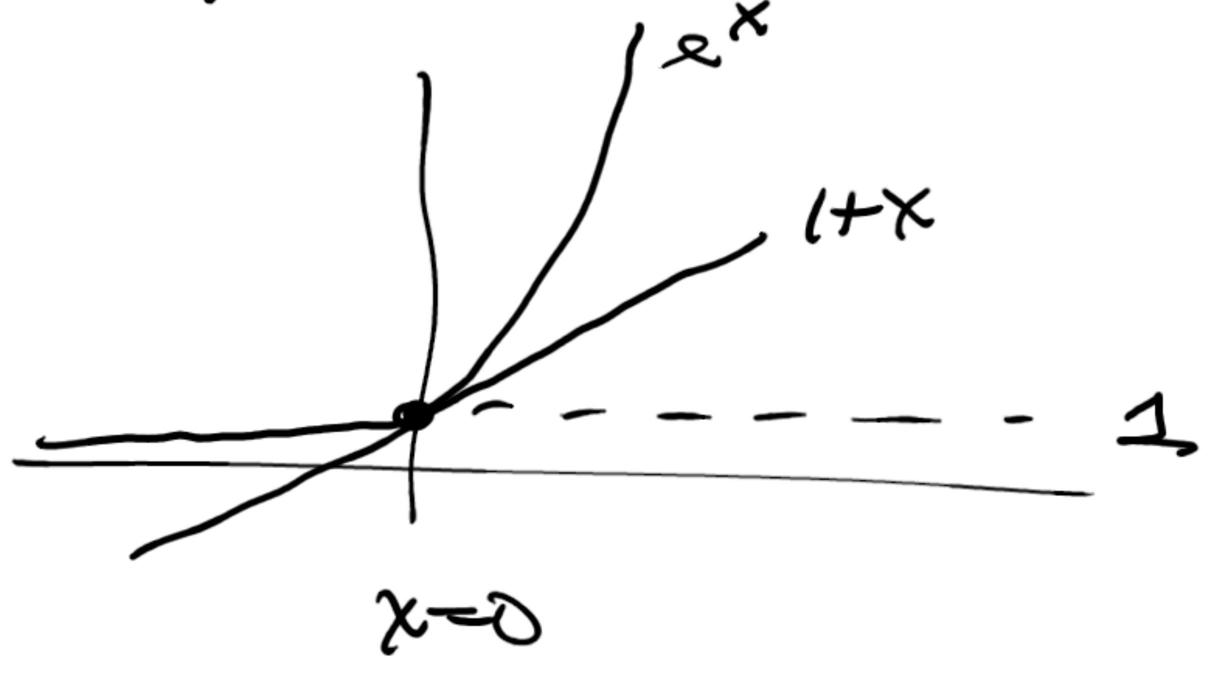
- exact expression are often difficult to obtain
- ______________________ to utilize

eg: ______ vs $\frac{1}{100}$ vs $\frac{1}{\ln n}$

lem [calculus]:

$$\forall x \quad 1 + x \leq e^x$$

sketch: taylor expansion



$\square$

rmk: $1 + x \approx e^x$ at $x \approx 0$

cor: $\left(1 - \frac{1}{n} \right)^{n-1} \geq \frac{1}{e}$, $n > 1$

pf:

$$\frac{1}{\left(1 - \frac{1}{n}\right)^{n-1}} = \frac{1}{\left(\frac{n-1}{n}\right)^{n-1}} = \left(\frac{n}{n-1} \right)^{n-1}$$

$$= \Big(1 + \underbrace{\tfrac{1}{n-1}}_{x} \Big)^{n-1} \leq \left(e^{1/(n-1)} \right)^{n-1} = e$$

$\blacksquare$

cor: $\Pr[P_i \text{ succeeds in round } j]$

$$\geq \frac{1}{e \cdot n}$$

$\Rightarrow$ $Pr[P_i$ fails in rounds $1,\dots,t]$

$= \prod_{j=1}^{t} \underbrace{Pr[P_i \text{ fails in round } j]}_{\leq (1-1/en)}$

$\leq (1-1/en)^t$

$\leq (e^{-1/en})^t$

$= e^{-t/en}$

$\Rightarrow$ $Pr[P_i$ fails rounds $1,\dots,c\cdot en] \leq 1/e^c$

$\Rightarrow$ in $\Theta(n)$ rounds P_i succeeds w.p. $\geq 1-\frac{1}{e^{\Theta(1)}}$

$= 1-\Theta(1)$

$= \Theta(1)$

$\Rightarrow$ in $\Theta(n \ln n)$ rounds $\geq 1-\frac{1}{e^{\Theta(\ln n)}}$

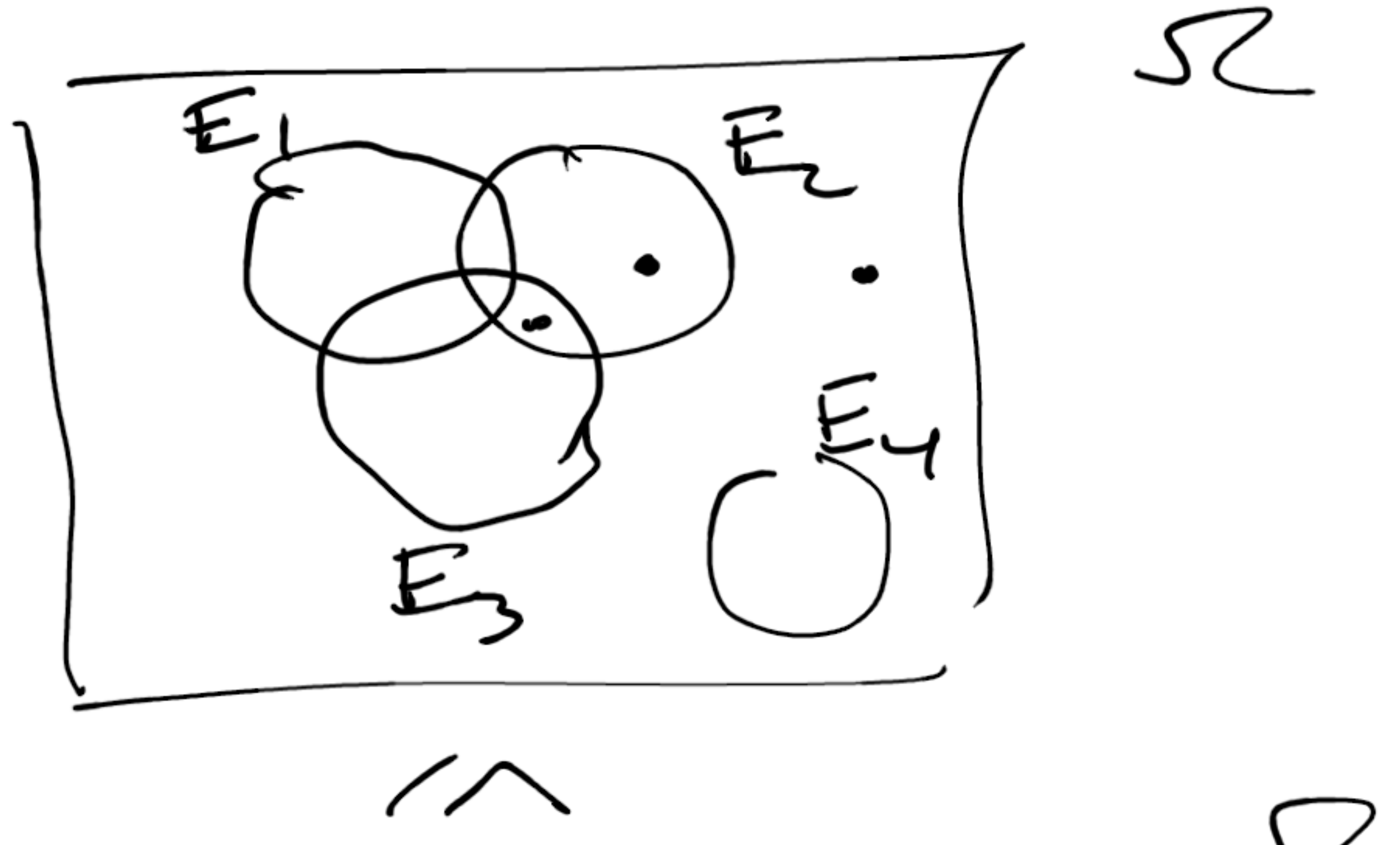
$= 1-\frac{1}{n^{\Theta(1)}}$

Q: $\boxed{\text{all}}$ players succeed?

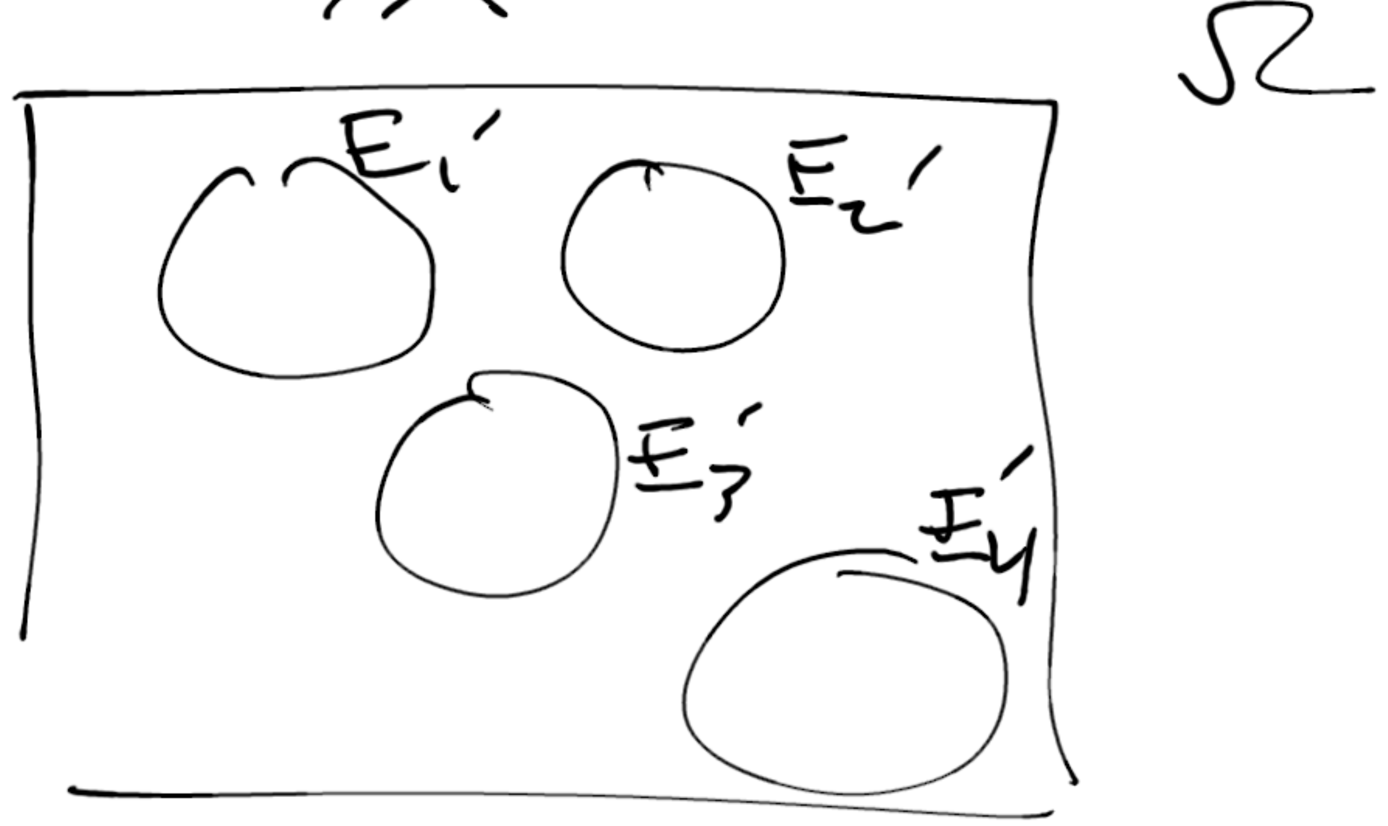
Lem [union bound]: events $E_1, \dots, E_n$

$$\Pr\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \Pr[E_i]$$

sketch:



vs



note: - n rounds clearly required
- can also analyze expected # rounds

Cor. $\Pr[\boxed{\text{any}}\ P_i \text{ fails in rounds } 1, \dots, c \cdot en]$

$$= \Pr\left[\bigcup_i P_i \text{ fails in } \dots\right]$$

$$\le \sum_{i=1}^{n} \underbrace{\Pr[P_i \text{ fails in } \dots]}_{\le 1/e^c}$$

$$\le n \cdot 1/e^c$$

Cor: $\Pr[\text{any } P_i \text{ fails in rounds } 1, \dots, (c+1) \cdot \ln n \cdot en] \le 1/n^c$

$\Rightarrow \Pr[\boxed{\text{all}}\ P_i \text{ communicate in first } \underbrace{2}_{\substack{1+1 \\ c=1}} en \ln n \text{ rounds}] \ge 1 - \frac{1}{n}$

today: randomized algo

- mot[illegible]tion
- def
- conversion [illegible]

next lecture: rand algo

logistics: pset 5 out F17