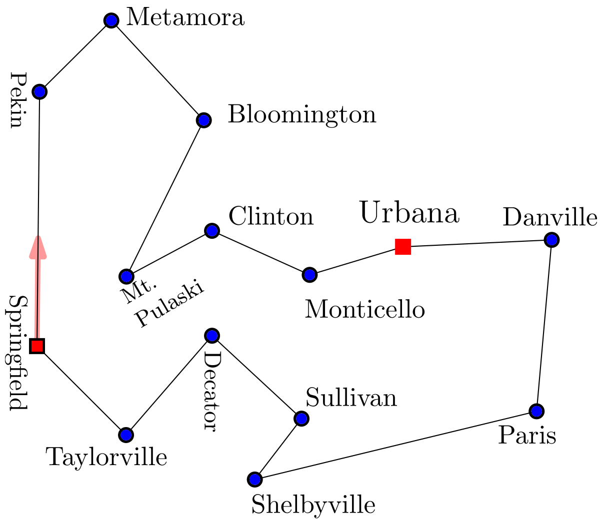
Approximation Algorithms for TSP

Lecture 24

May 4, 2021

Most slides are courtesy Prof. Chekuri

Lincoln's Circuit Court Tour



Traveling Salesman/Salesperson Problem (TSP)

Perhaps the most famous discrete optimization problem

Input: A graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}_+$.
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Assumption for simplicity: Graph $G = (V, E)$ is a complete graph. Can add missing edges with infinite cost to make graph complete.

Observation: Once graph is complete there is always a Hamiltonian cycle but only Hamiltonian cycles of finite cost are Hamiltonian cycles in the original graph.

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  - Construct a complete graph G' on vertices V, with cost 1 on edges of G and ∞ on all other edges.

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  - Suppose, $G = (V, E)$ is a simple graph in which we want to find a Hamiltonian cycle.
  - Construct a complete graph G' on vertices V, with cost 1 on edges of G and ∞ on all other edges.
  - If G has a Hamiltonian cycle then there is a TSP tour of cost n in G', else the cost is ∞.

Important Special Cases

Metric-TSP: $G = (V, E)$ is a **complete graph** and c defines a metric space. $c(u, v) = c(v, u)$ for all u, v and $c(u, w) \leq c(u, v) + c(v, w)$ for all u, v, w.

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Geometric-TSP: V is a set of points in some Euclidean d-dimensional space $\mathbb{R}^d$ and the distance between points is defined by some norm such as standard Euclidean distance, L_1/Manhatta distance etc.

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Metric-TSP is NP-Hard.

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Given $G = (V, E)$ we create a new complete graph $G' = (V, E')$ with the following costs. If $e \in E$ cost $c(e) = 1$. If $e \in E' - E$ cost $c(e) = 2$.

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Given $G = (V, E)$ we create a new complete graph $G' = (V, E')$ with the following costs. If $e \in E$ cost $c(e) = 1$. If $e \in E' - E$ cost $c(e) = 2$. Easy to verify that c satisfies metric properties. Moreover, G' has TSP tour of cost n iff G has a Hamiltonian Cycle. □

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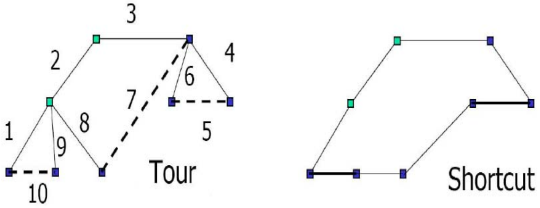
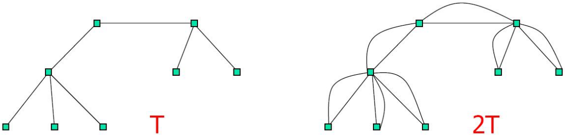
Because, any such tour can be converted in to a simple cycle of smaller cost by adding “short-cuts”.



Approximation for Metric-TSP

MST-Heuristic$(G = (V, E), c)$

```
Compute a minimum spanning tree (MST) T in G
Obtain an Eulerian graph H = 2T by doubling edges of T
An Eulerian tour of H gives a tour of G
Obtain Hamiltonian cycle by shortcutting the tour
```



Analyzing MST-Heuristic

Lemma

Let $c(T) = \sum_{e \in T} c(e)$ *be cost of MST. We have* $c(T) \leq OPT$.

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Proof.

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Theorem

MST-Heuristic gives a 2-approximation for Metric-TSP.

Proof.

Cost of tour is at most $2c(T)$ and taking shortcuts only reduces the cost due to triangle ineuqlity. Hence MST-Heuristic gives a 2-approximation. □

Question

Consider the subgraph induced by edges of a tour that visits every vertex at least once (a closed walk). The degree of every vertex in this subgraph is:

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Euler Tour!

Background on Eulerian graphs

Definition

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An undirected multigraph $G = (V, E)$ *is Eulerian iff* G *is connected and every vertex degree is even.*

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Theorem

A directed multigraph $G = (V, E)$ is Eulerian iff G is weakly connected and for each vertex v, $\mathit{indeg}(v) = \mathit{outdeg}(v)$.

Improved approximation for Metric-TSP

How can we improve the MST-heuristic?

Observation: Finding optimum TSP tour in G is same as finding minimum cost Eulerian subgraph of G (allowing duplicate copies of edges).

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    Compute a minimum spanning tree (MST) T in G
    Add edges to T to make Eulerian graph H
    An Eulerian tour of H gives a tour of G
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How do we add edges to make T Eulerian?

Christofides Heuristic: 3/2 approximation

Christofides-Heuristic$(G = (V, E), c)$

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Christofides Heuristic: $3/2$ approximation

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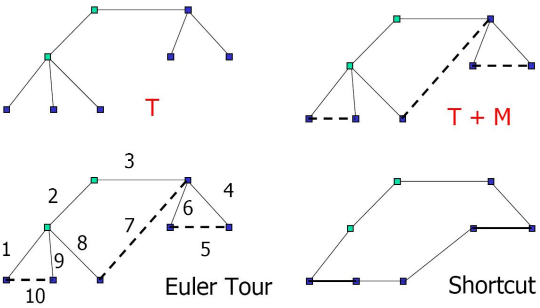

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Main lemma:

Lemma

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Assuming lemma:

Theorem

Christofides heuristic returns a tour of cost at most $3OPT/2$.

Proof.

$c(H) = c(T) + c(M) \leq OPT + OPT/2 \leq 3OPT/2$. Cost of tour is at most cost of H. □

Analysis of Christofides Heuristic

Lemma

Suppse $G = (V, E)$ *is a metric and* $S \subset V$ *be a subset of vertices. Then there is a TSP tour in* $G[S]$ *(the graph induced on* S*) of cost at most* OPT.

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Proof.

Let $C = v_1, v_2, \ldots, v_n, v_1$ be an optimum tour of cost OPT in G and let $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ where, without loss of generality $i_1 < i_2 \ldots < i_k$. Then consider the tour $C' = v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_1}$ in $G[S]$. The cost of this tour is at most cost of C due to shortcutting. □

Proof of lemma for Christofides heuristic

Lemma

$c(M) \leq OPT/2$.

Recall that M is a matching on S the set of odd degree nodes in T. Recall that $|S|$ is even. Let $S = \{v_1, v_2, \ldots, v_{2k}\}$.

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Proof.

From previous lemma, there is tour of cost OPT for S in $G[S]$. Wlog let this tour be $v_1, v_2, \dots, v_{2k}, v_1$. Consider two matchings M_a and M_b where $M_a = \{(v_1, v_2), (v_3, v_4), \dots, (v_{2k-1}, v_{2k})$ and $M_b = \{(v_2, v_3), (v_4, v_5), \dots, (v_{2k}, v_1)$.
$M_a \cup M_b$ is set of edges of the tour so $c(M_a) + c(M_b) \leq OPT$.

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$M_a \cup M_b$ is set of edges of the tour so $c(M_a) + c(M_b) \leq OPT$.
$c(\text{min-cost matching}) \leq \min\{c(M_a), c(M_b)\} \leq OPT/2$. □

Other comments

Christofides algorithm came in 1976, and was not improved until 2020!
[Karlin, Klein, Gharan'20] Gave $(3/2 - \epsilon)$ approximation, for some $\epsilon > 10^{-36}$.

Major open problem in approximation algorithms.

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For points in any fixed dimension d there is a polynomial-time approximation scheme. For any fixed $\epsilon > 0$ a tour of cost $(1 + \epsilon)OPT$ can be computed in polynomial time. [Arora 1996, Mitchell 1996].

Excellent practical code exists for solving large scale instances of TSP that arise in several applications. See Concorde TSP Solver by Applegate, Bixby, Chvatal, Cook.

Directed Graphs and Asymmetric TSP (ATSP)

Question: What about directed graphs?

Equivalent of Metric-TSP is Asymmetric-TSP (ATSP)

- Input is a complete directed graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}_+$.
- Edge costs are not necessarily symmetric. That is $c(u, v)$ can be different from $c(v, u)$
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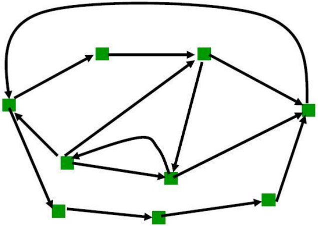

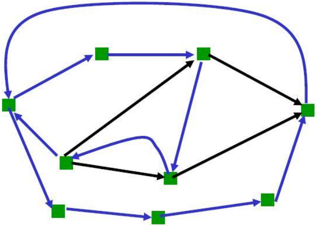

Same as finding a minimum cost connected Eulerian subgraph of G.

Approximation for ATSP

Harder than Metric-TSP

- Simple $\log_2 n$ approximation from 1980.
- Improved to $O(\log n / \log \log n)$-approximation in 2010.
- Further improved to $O((\log \log n)^c)$-approximation in 2015.
- Finally to c-approximation in 2018, where $c = 5500$!

Believed that the constant should be much smaller. Lower bound is 2 for LP relaxations.

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Ans: Reduces to minimum cost bipartite matching!

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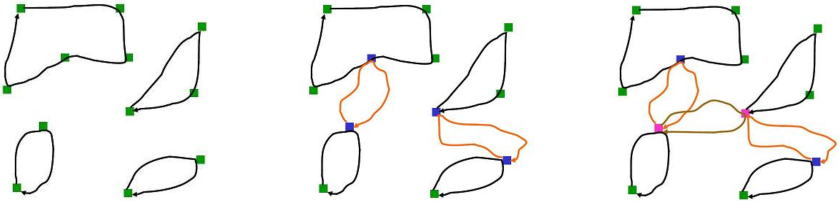
    Shortcut C' to obtain a cycle on V and output C'.

Illustration and Analysis



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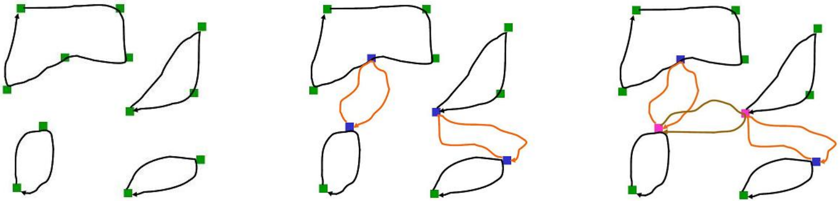

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Lemma

The number of vertices shrinks by half in each iteration and hence total of at most $\lceil \log n \rceil$ *cycle covers.*

Hence total cost of all cycle covers is at most $\lceil \log n \rceil \cdot OPT$.