CS 473: Algorithms

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LP & Strong Duality

Lecture 19
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Some of the slides are courtesy Prof. Chekuri

Part I

Recall

Linear Program

Properties and the Simplex Algorithm

- Solution at a vertex of the polyhedron $\mathcal{P}$.
- If vertex v is not optimal then it has a neighbor where the objective value improves.
- If the $\mathcal{P}$ in d dimension, then every vertex has exactly d neighboring vertices (almost always).

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Simplex: Moves from a vertex to its neighboring vertex

Questions + Answers

- Which neighbor to move to? One where objective value increases.

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- Which neighbor to move to? One where objective value increases.
- When to stop? When no neighbor with better objective value.
- How much time does it take? At most d neighbors to consider in each step.

Issues

1. Starting vertex
2. The linear program could be infeasible: No point satisfy the constraints.
3. The linear program could be unbounded: Polygon unbounded in the direction of the objective function.

Computing the Starting Vertex

Equivalent to solving another LP!

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s.t. \quad & \textstyle\sum_j a_{ij} x_j - s \leq b_i, \quad \forall i \\
& s \geq 0
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Trivial feasible solution:

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If $Ax \leq b$ feasible then optimal value of the above LP is $s = 0$.

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Checks Feasibility!

Part II

Duality

Feasible Solutions and Lower Bounds

Consider the program

$$
\begin{array}{llrcl}
\text{maximize} & 4x_1 + & 2x_2 & & \\
\text{subject to} & x_1 + & 3x_2 & \leq & 5 \\
 & 2x_1 - & 4x_2 & \leq & 10 \\
 & x_1 + & x_2 & \leq & 7 \\
 & x_1 & & \leq & 5
\end{array}
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1. $(0, 1)$ satisfies all the constraints and gives value 2 for the objective function.
2. Thus, optimal value σ^* is at least 2.
3. $(2, 0)$ also feasible, and gives a better bound of 8.
4. How good is 8 when compared with σ^*?

Obtaining Upper Bounds

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$$\begin{array}{rrrl} 2(& x_1 + & 3x_2 &) \leq 2(5) \\ +1(& 2x_1 - & 4x_2 &) \leq 1(10) \\ \hline & 4x_1 + & 2x_2 & \leq 20 \end{array}$$

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2. Thus, 20 is an upper bound on the optimum value!

1. Multiply first equation by y_1, second by y_2, third by y_3 and fourth by y_4 $(y_1, y_2, y_3, y_4 \geq 0)$ and add

$$
\begin{array}{rrrl}
y_1(& x_1 + & 3x_2 &) \leq y_1(5) \\
+y_2(& 2x_1 - & 4x_2 &) \leq y_2(10) \\
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\multicolumn{4}{l}{(y_1 + 2y_2 + y_3 + y_4)x_1 + (3y_1 - 4y_2 + y_3)x_2 \leq \ldots}
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Generalizing . . .

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2. $5y_1 + 10y_2 + 7y_3 + 5y_4$ is an upper bound, provided coefficients of x_i are same as in the objective function $(4x_1 + 2x_2)$,

$$y_1 + 2y_2 + y_3 + y_4 = 4 \quad 3y_1 - 4y_2 + y_3 = 2$$

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3. Subject to these constrains, the best upper bound is $\min : 5y_1 + 10y_2 + 7y_3 + 5y_4$!

Dual LP: Example

Thus, the optimum value of program

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \leq 5 \\ & 2x_1 - 4x_2 \leq 10 \\ & x_1 + x_2 \leq 7 \\ & x_1 \leq 5 \end{aligned}$$

is upper bounded by the optimal value of the program

$$\begin{aligned} \text{minimize} \quad & 5y_1 + 10y_2 + 7y_3 + 5y_4 \\ \text{subject to} \quad & y_1 + 2y_2 + y_3 + y_4 = 4 \\ & 3y_1 - 4y_2 + y_3 = 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Dual Linear Program

Given a linear program $\mathbf{\Pi}$ in canonical form

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots n \end{array}$$

the dual $\mathbf{Dual}(\mathbf{\Pi})$ is given by

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Dual Linear Program

Given a linear program Π in canonical form

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the dual $\mathrm{Dual}(\Pi)$ is given by

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Proposition

$\mathrm{Dual}(\mathrm{Dual}(\Pi))$ *is equivalent to* Π

Duality Theorems

Theorem (Weak Duality)

If x' *is a feasible solution to* Π *and* y' *is a feasible solution to* $\mathrm{Dual}(\Pi)$ *then* $c \cdot x' \leq y' \cdot b$.

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Theorem (Strong Duality)

If x^* *is an optimal solution to* Π *and* y^* *is an optimal solution to* $\mathrm{Dual}(\Pi)$ *then* $c \cdot x^* = y^* \cdot b$.

Many applications! Maxflow-Mincut theorem can be deduced from duality.

Weak Duality

Theorem (Weak Duality)

If x is a feasible solution to Π and y is a feasible solution to $\mathrm{Dual}(\Pi)$ then $c \cdot x \leq y \cdot b$.

We already saw the proof by the way we derived it but we will do it again formally.

Proof.

Since y' is feasible in $\mathrm{Dual}(\Pi)$: $y'A = c$

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Therefore $c \cdot x' = y'Ax'$

Since x' is feasible in Π, $Ax' \leq b$ and hence,

$$c \cdot x' = y'Ax' \leq y' \cdot b$$

Strong Duality and Complementary Slackness

$$\begin{array}{ll} \textbf{maximize}: & c \cdot x \\ \text{subject to} & Ax \leq b \end{array} \quad \xrightarrow{\textit{Dual}} \quad \begin{array}{ll} \textbf{minimize}: & y \cdot b \\ \text{subject to} & yA = c \\ & y \geq 0 \end{array}$$

Definition (Complementary Slackness)

x feasible in Π and y feasible in $\mathrm{Dual}(\Pi)$, s.t.,

$$\forall i = 1..n, \quad y_i > 0 \;\Rightarrow\; (Ax)_i = b_i$$

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Geometric Interpretation: c is in the cone of the normal vectors of the tight hyperplanes at x.

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Theorem

(x^*, y^*) satisfies complementary Slackness if and only if strong duality holds, i.e., $c \cdot x^* = y^* \cdot b$.

Proof.

$$\begin{aligned} c \cdot x^* &= (y^* A) \cdot x^* \\ &= y^* \cdot (Ax^*) \end{aligned}$$

$(\Rightarrow)$

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□

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$(\Leftarrow)$ **Exercise** □

Strong Duality ≡ Complementary Slackness

We want

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Strong Duality $\equiv$ Complementary Slackness

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Theorem

(x^*, y^*) satisfies complementary slackness $\Leftrightarrow$ Strong duality holds, i.e., $c \cdot x^* = y^* \cdot b$.

If (x^*, y^*) optimum $\Rightarrow$ complementary slackness, then done.

Complementary Slackness: Geometric View

$$\begin{array}{ll} \textbf{maximize}: & c \cdot x \\ \text{subject to} & Ax \le b \end{array} \quad \xrightarrow{\textit{Dual}} \quad \begin{array}{ll} \textbf{minimize}: & y \cdot b \\ \text{subject to} & yA = c \\ & y \ge 0 \end{array}$$

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Suppose first d inequalities are tight at x^*.

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x^* **:** Optimum vertex.

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$$Ax^* \leq b \quad \text{splits into} \quad \hat{A}x^* = \hat{b}, \qquad \tilde{A}x^* < \tilde{b}$$

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Suppose $\boldsymbol{c}$ is NOT in the cone of rows of $\boldsymbol{\hat{A}}$.

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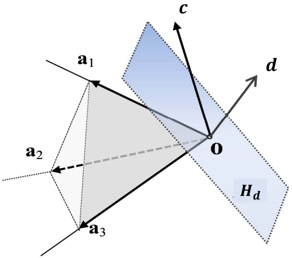

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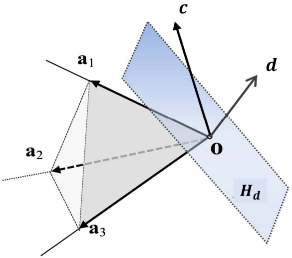

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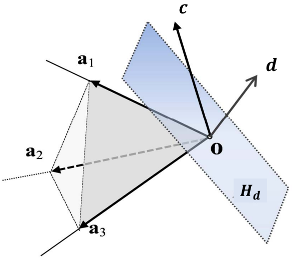

$\Rightarrow$ There exists a hyperplane separating c from the cone.
Suppose cone is on the negative side, and c on the positive size. If the d is the normal vector of the hyperplane, then formally,

$$\hat{A}d < 0, \qquad c \cdot d > 0$$

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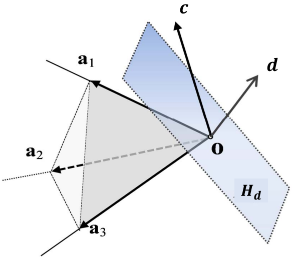

$\Rightarrow$ There exists a hyperplane separating $\boldsymbol{c}$ from the cone.
Suppose cone is on the negative side, and $\boldsymbol{c}$ on the positive size. If the $\boldsymbol{d}$ is the normal vector of the hyperplane, then formally,

$$\hat{\boldsymbol{A}}\boldsymbol{d} < \boldsymbol{0}, \qquad \boldsymbol{c} \cdot \boldsymbol{d} > \boldsymbol{0} \qquad \text{Also known as Farkas' Lemma}$$

Optimality implies Complementary Slackness

x^* **:** Optimum vertex.

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$\boldsymbol{c} \cdot (\boldsymbol{x}^* + \epsilon \boldsymbol{d}) = \boldsymbol{c} \cdot \boldsymbol{x}^* + \epsilon(\boldsymbol{c} \cdot \boldsymbol{d}) > \boldsymbol{c} \cdot \boldsymbol{x}^* \Rightarrow \boldsymbol{x}^*$ **is NOT optimum!**

Proof of Strong Duality

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$$Ax^* \leq b \quad \text{splits into} \quad \hat{A}x^* = \hat{b}, \qquad \tilde{A}x^* < \tilde{b}$$

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$\Leftrightarrow$ $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ satisfies strong duality, $\boldsymbol{c} \cdot \boldsymbol{x}^* = \boldsymbol{y}^* \cdot \boldsymbol{b}$. (implies $\boldsymbol{y}^*$ optimal in $\mathbf{Dual}(\boldsymbol{\Pi})$).

Theorem (Strong Duality)

If x^* is an optimal solution to Π and y^* is an optimal solution to $\mathbf{Dual}(\Pi)$ then $c \cdot x^* = y^* \cdot b$.

Duality for another canonical form

Compactly, for the primal LP Π

$$\begin{array}{ll} \max & c \cdot x \\ \text{subject to} & Ax \leq b, \; x \geq 0 \end{array}$$

the dual LP is $\mathrm{Dual}(\Pi)$

$$\begin{array}{ll} \min & y \cdot b \\ \text{subject to} & yA \geq c, \; y \geq 0 \end{array}$$

Definition (Complementary Slackness)

x feasible in Π and y feasible in $\mathrm{Dual}(\Pi)$, s.t.,

$$\forall i = 1, \ldots, n, \quad y_i > 0 \; \Rightarrow \; (Ax)_i = b_i$$
$$\forall j = 1, \ldots, d, \quad x_j > 0 \; \Rightarrow \; (yA)_j = c_j$$

In General...

from Jeff's notes

Primal	Dual
$\max c \cdot x$	$\min y \cdot b$
$\sum_j a_{ij} x_j \leq b_i$	$y_i \geq 0$
$\sum_j a_{ij} x_j \geq b_i$	$y_i \leq 0$
$\sum_j a_{ij} x_j = b_i$	–
$x_j \geq 0$	$\sum_i y_i a_{ij} \geq c_j$
$x_j \leq 0$	$\sum_i y_i a_{ij} \leq c_j$
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Figure H.4. Constructing the dual of an arbitrary linear program.

Part III

Examples of Duality

Network flow

s-t flow in directed graph $G = (V, E)$ with capacities c. Assume for simplicity that no incoming edges into s.

$$
\begin{aligned}
\textbf{max} \quad & \sum_{(s,v)\in \mathrm{E}} x(s,v) \\
& \sum_{(u,v)\in \mathrm{E}} x(u,v) - \sum_{(v,w)\in \mathrm{E}} x(v,w) = 0 \quad \forall v \in \mathrm{V} \setminus \{s,t\} \\
& x(u,v) \leq c(u,v) \qquad \forall (u,v) \in \mathrm{E} \\
& x(u,v) \geq 0 \qquad \forall (u,v) \in \mathrm{E}.
\end{aligned}
$$

Network flow: Equivalent formulation

Directed graph $G = (V, E)$, with capacities on edges. Add a t to s edge with infinite capacity. Now maximize flow on this edge.

$$\begin{aligned}
\textbf{max} \quad & x(t, s) \\
& \sum_{(u,v)\in \mathrm{E}} x(u, v) - \sum_{(v,w)\in \mathrm{E}} x(v, w) = 0 && \forall v \in V \\
& x(u, v) \leq c(u, v) && \forall (u, v) \in \mathrm{E} \setminus (t, s) \\
& x(u, v) \geq 0 && \forall (u, v) \in \mathrm{E}.
\end{aligned}$$

Dual of Network Flow

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4. Duality is a critical tool in the theory of linear programming. Duality implies the Linear Programming is in **co-NP**. Do you see why?

Part IV

Integer Linear Programming

Integer Linear Programming

Problem

Find a vector $x \in Z^d$ (integer values) that

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for } i = 1 \dots n \end{array}$$

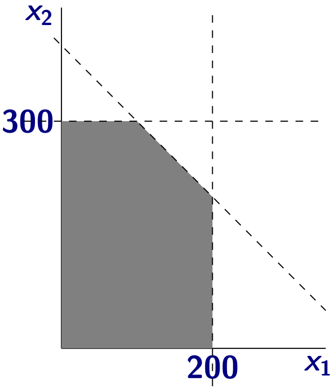
Input is matrix $A = (a_{ij}) \in \mathbb{R}^{n \times d}$, column vector $b = (b_i) \in \mathbb{R}^n$, and row vector $c = (c_j) \in \mathbb{R}^d$

Factory Example

$$\begin{array}{ll} \text{maximize} & x_1 + 6x_2 \\ \text{subject to} & x_1 \leq 200 \quad x_2 \leq 300 \quad x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{array}$$

Suppose we want x_1, x_2 to be integer valued.

Factory Example Figure



1. Feasible values of x_1 and x_2 are integer points in shaded region
2. Optimization function is a line; moving the line until it just leaves the final integer point in feasible region, gives optimal values

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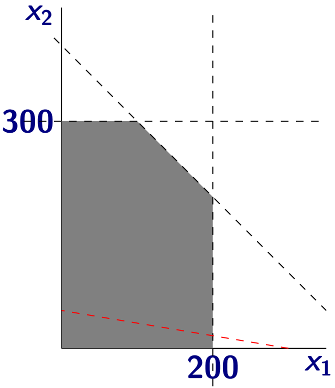

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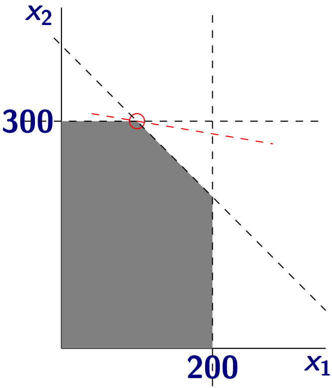

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Practice: integer programs are solved by a variety of methods

1. branch and bound
2. branch and cut
3. adding cutting planes
4. linear programming plays a fundamental role

Example: Maximum Independent Set

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in S. That is, if $u, v \in S$ then $(u, v) \notin E$.

Input Graph $G = (V, E)$

Goal Find maximum sized independent set in G

Example: Dominating Set

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is a dominating set if for all $v \in V$, either $v \in S$ or a neighbor of v is in S.

Input: Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal: Find minimum weight dominating set in G

Example: s-t minimum cut and implicit constraints

Input Graph $G = (V, E)$, edge capacities $c(e), e \in E$.
$s, t \in V$

Goal Find minimum capacity s-t cut in G.

Linear Programs with Integer Vertices

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Luck or *Structure*:

1. Linear program for flows with integer capacities have integer vertices
2. Linear program for matchings in bipartite graphs have integer vertices
3. A complicated linear program for matchings in general graphs have integer vertices.

All of above problems can hence be solved efficiently.

Linear Programs with Integer Vertices

Meta Theorem: A combinatorial optimization problem can be solved efficiently if and only if there is a linear program for problem with integer vertices.

Consequence of the Ellipsoid method for solving linear programming.

In a sense linear programming and other geometric generalizations such as convex programming are the most general problems that we can solve efficiently.

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4. Duality is a critical tool in the theory of linear programming. Duality implies the Linear Programming is in **co-NP**. Do you see why?
5. **Integer Programming in NP-Complete. LP-based techniques critical in heuristically solving integer programs.**