Flow Variants

Lecture 16

March 30, 2021

Most slides are courtesy Prof. Chekuri

Generalizations of Flow

We have seen s-t flow. Flow problems admit several generalizations and variations.

- Demands and Supplies (we have already seen them)
- Circulations
- Lower bounds in addition to upper bounds
- Minimum cost flows and circulations
- Flows with losses
- Flows with time delays
- Multi-commodity flows
- $\cdots$

Many applications, connections, algorithms.

Part I

Circulations

Circulations

Definition

Circulation in a network $G = (V, E)$, is function $f : E \to \mathbb{R}^{\geq 0}$ s.t.

1. Conservation Constraint: For each vertex v:

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

2. Capacity Constraint: For each edge e, $f(e) \leq c(e)$

No source or sink. $f(e) = 0$ for all e is a valid circulation.

Circulation with lower bounds

Circulations are useful mainly in conjunction with *lower bounds*.
Given a network $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ with *capacities* $c : E \to \mathbb{R}^{\geq 0}$ and *lower bounds* $\ell : E \to \mathbb{R}^{\geq 0}$.

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3. Lower bound Constraint: For each edge e, $f(e) \geq \ell(e)$

Circulation problem

Problem

Input A network G with capacity c and lower bound ℓ

Goal Find a feasible circulation

Simply a feasibility problem.

Observation: As hard as the s-t maxflow!

Reducing Max-flow to Circulation

Decision version of max-flow.

Problem

Input: A network G with capacity c and source s and sink t and number F.

Goal: Is there an s-t flow of value at least v in G?

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1. set $\ell(e) = 0$ for each e in G
2. add new edge (t, s) with lower bound v and upper bound ∞

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Claim

There exists a flow of value v from s to t in G if and only if there exists *a feasible circulation in G'.*

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Important properties of circulations:

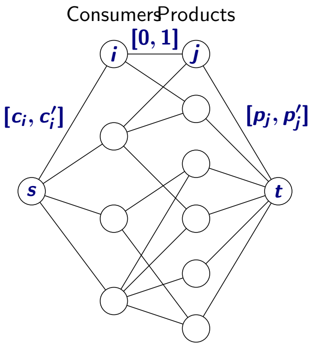
- Reduction shows that one can find in $O(mn)$ time a feasible circulation in a network with capacities and lower bounds
- If edge capacities and lower bounds are integer valued then there is always a feasible integer-valued circulation
- Hoffman's circulation theorem is the equivalent of maxflow-mincut theorem.
- Circulation can be decomposed into at most m cycles in $O(mn)$ time.

Survey Design

Application of Circulations

1. Design survey to find information about n_1 products from n_2 customers.
2. Can ask customer questions only about products purchased in the past.
3. Customer can only be asked about at most c'_i products and at least c_i products.
4. For each product need to ask at east p_i consumers and at most p'_i consumers.

Reduction to Circulation



1. include edge (i, j) is customer i has bought product j
2. Add edge (t, s) with lower bound 0 and upper bound ∞.
   1. Consumer i is asked about product j if the integral flow on edge (i, j) is 1

Part II

Minimum Cost Flows

Minimum Cost Flows

1. **Input:** Given a flow network G and also edge costs, $w(e)$ for edge e, and a flow requirement F.
2. **Goal:** Find a *minimum cost* flow of value F from s to t
3. **Goal:** Find a *minimum cost* maximum s-t flow

Given flow $f : E \to R^+$, cost of flow $= \sum_{e \in E} w(e) f(e)$.

Note: costs can be negative. An optimum solution may need cycles.

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Much more general than the shortest path problem.

Minimum Cost Flow: Facts

1. problem can be solved efficiently in polynomial time
   1. $O(nm \log C \log(nW))$ time algorithm where C is maximum edge capacity and W is maximum edge cost
   2. $O(m \log n(m + n \log n))$ time strongly polynomial time algorithm
2. for integer capacities there is always an optimum solution in which flow is integral

Min-Cost Flow: Residual Graphs

Residual graph when there are costs:

Definition

For a network $G = (V, E)$ and flow f, the **residual graph** $G_{f,w} = (V', E')$ of G with respect to f and w is

1. $V' = V$,
2. **Forward Edges**: For each edge $e \in E$ with $f(e) < c(e)$, we add $e \in E'$ with capacity $c(e) - f(e)$. Cost $w'(e) = w(e)$.
3. **Backward Edges**: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we add $(v, u) \in E'$ with capacity $f(e)$. Cost $w'(e) = -w(e)$.

Min-Cost Flow: Optimality Condition

Question: Suppose f is a max s-t flow in G. When is f a min-cost a minimum cost max-flow?

Min-Cost Flow: Optimality Condition

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If and only if there is *no negative-cost cycle* in G_f.

- If there is a negatice cost cycle we can augment along the cycle and reduce the cost of f (note that value of f does not change)
- Suppose f' is another maxflow of less cost. One can show that $f' - f$ is a circulation in G_f (since both are maxflows) which means that $f' - f$ can be decomposed into cycles. Since f' has less cost than f there must be a negative cost cycle.

Min-Cost Flwo: Cycle-canceling algorithm

Goal: Given G with integer capacities, non-negative weights, find s-t maxflow of with minimum cost.

Cycle-Canceling-Alg

```
Compute a maxflow f in G (ignoring costs)
G_{f,w} is residual graph of G with respect to f
while there is a negative weight cycle C in G_{f,w} do
    let C be a negative weight cycle in G_{f,w}
    Augment one unit of flow along C and update f
    Construct new residual graph G_{f,w}.
Output f
```

Like Ford-Fulkerson the run-time is pseudo-polynomial in costs. Can be implemented to run in $O(m^2 nCW)$ time where $C = \max_e c(e)$ and $W = \max_e |w(e)|$.

Min-Cost Flow: Successive Shortest Path Alg

Goal: Given G with integer capacities, **non-negative** weights, and integer k, find s-t flow of value k with minimum cost.

```
Successive-Shortest-Path-Alg
    for every edge e, f(e) = 0
    G_{f,w} is residual graph of G with respect to f
    while v(f) < k and G_{f,w} has a simple s-t path do
        let P be a shortest s-t path in G_{f,w}
        Augment one unit of flow along P and update f
        Construct new residual graph G_{f,w}.
```

Algorithm gives optimum solution. Shows existence of integral optimum solution for integer capacities. Run time is $O(mk \log m)$, and in the worst-case, $O(mC \log m)$.

Maximum Profit Flow?

Can we find find a maxflow of maximum profit?