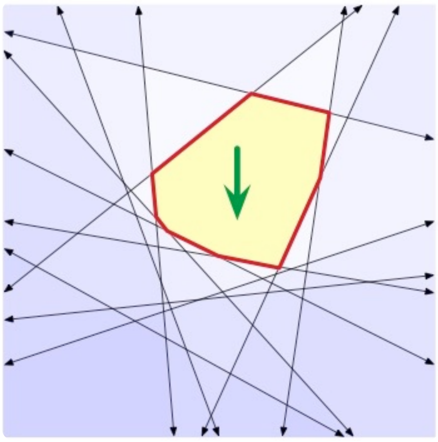


LP = find lowest point in convex polyhedron

$\iff$

$\mathbb{R}^n$

?

n+2 constraints

d variables

n constraints

d constraints

Primal (Π)

$$\begin{array}{ll} \max & c \cdot x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array}$$

$\iff$

Dual (∐)

$$\begin{array}{ll} \min & y \cdot b \\ \text{s.t.} & yA \ge c \\ & y \ge 0 \end{array}$$

n variables

d constraints

n constraints

The Fundamental Theorem of Linear Programming. *A canonical linear program Π has an optimal solution x^* if and only if the dual linear program ∐ has an optimal solution y^* such that* $\boxed{c \cdot x^* = y^* A x^* = y^* \cdot b.}$

optimal objective value

Weak duality: If x is feasible for Π
y is feasible for ∐

Then $c \cdot x \le yAx \le y \cdot b$

Proof: x is feasible $\Rightarrow$ $Ax \le b$
y is feasible $\Rightarrow$ $y \ge 0$ $\Rightarrow$ $yAx \le y \cdot b$

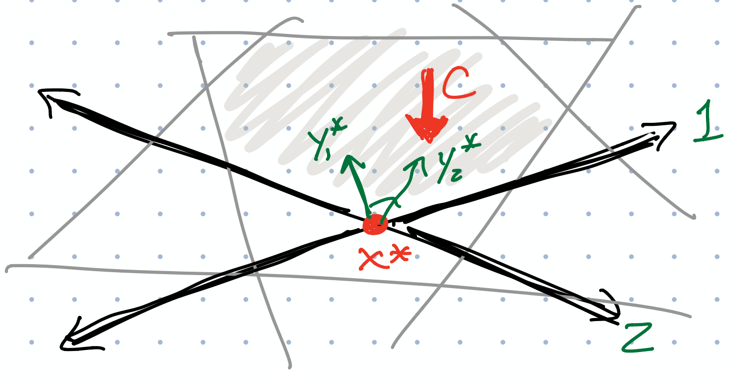
Symmetry $\Rightarrow$ $cx \le yAx$ □

$\mathbb{R}$

$\leftarrow \{y \cdot b \mid yA \ge c, y \ge 0\}$

Strong duality $\Rightarrow$ no gap

$\leftarrow \{c \cdot x \mid Ax \le b, x \ge 0\}$

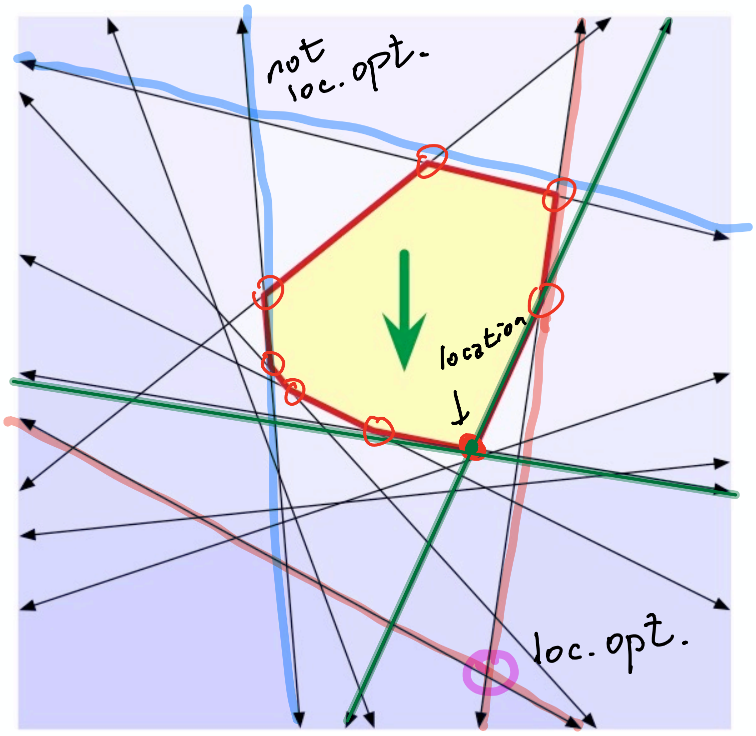


Dual variables y^*

Normals to opt. constraints
define a coordinate frame

Dual variables = coeffs of c
in this coord frame

y_1^* = force applied by constraint 1
y_2^* = ——————————— 2



basis = set of d ~~linearly independent~~ constraints

(ignore degeneracies)

location $\longrightarrow$ solve equations

value of a basis = c · location

There are exactly $\binom{n+d}{d}$ bases

Basis is feasible if $Ax \le b$, $x \ge 0$

where x = location

$\Longleftrightarrow$ vertex of feasible polyhedron

Basis is locally optimal

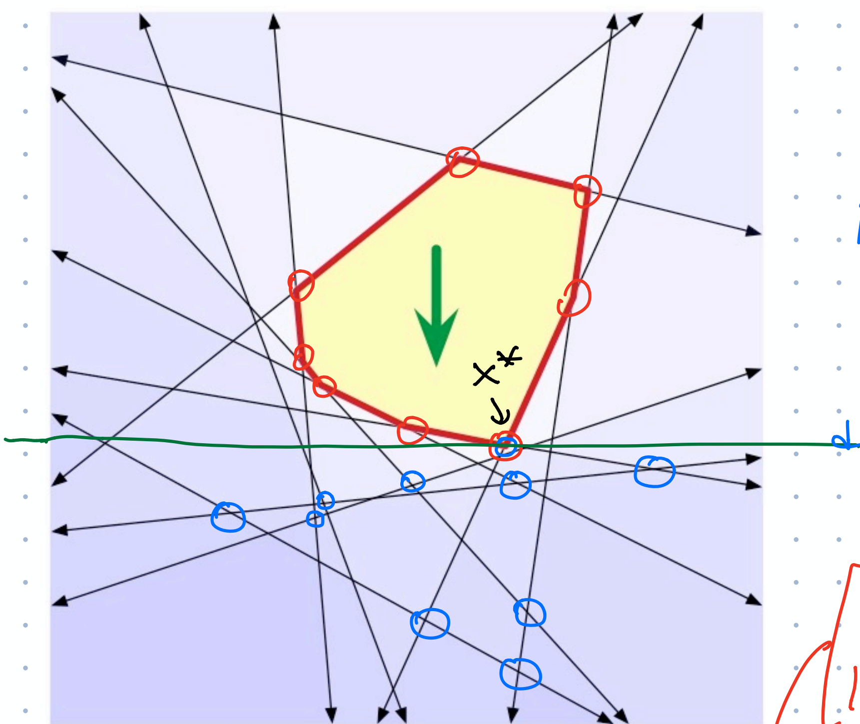
$\Leftrightarrow$ location is optimal for LP with same obj and only constraints in the basis

$$\boxed{\begin{array}{ll} \max & c \cdot x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array}}$$

— d variables
> n+d constraints

$$\boxed{\begin{array}{ll} \min & y \cdot b \\ \text{s.t.} & yA \ge c \\ & y \ge 0 \end{array}}$$

— n vars
> d+n constraints



$c \cdot x \le y \cdot b$

$c \cdot x^* = y^* \cdot b$

Feasible $\rightleftarrows$ vertices of feasible region

locally optimal $\longleftrightarrow$ dual feasible

basis $\rightleftarrows$ dual basis

d constraints $\quad$ n constraints

$\binom{n+d}{d} = \binom{d+n}{n}$

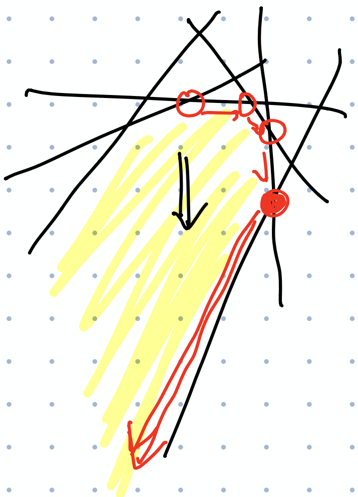
feasible $\rightleftarrows$ loc. opt.

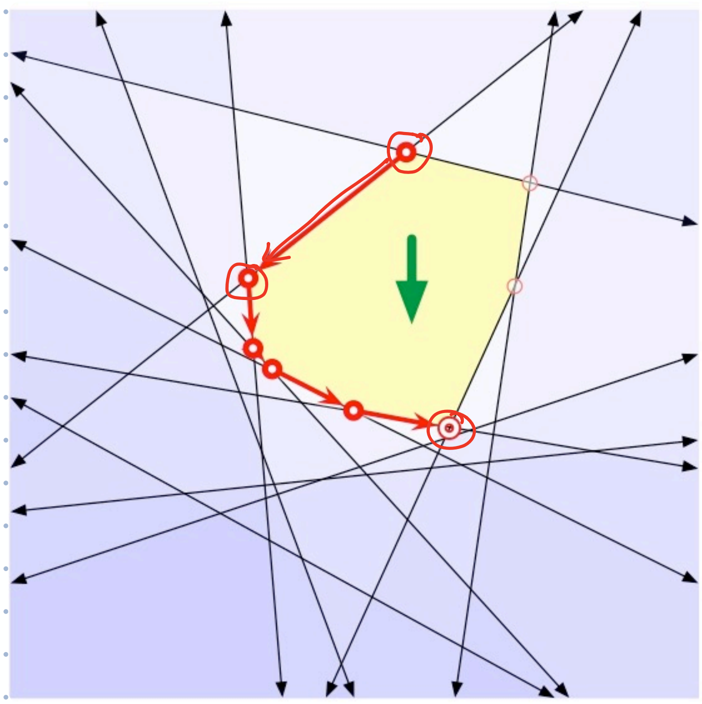
loc. opt $\rightleftarrows$ feasible

optimal $\rightleftarrows$ optimal

```
PRIMALSIMPLEX(H):
  if ∩H = ∅
      return INFEASIBLE
  x ← any feasible vertex / basis
  while x is not locally optimal
      ⟨⟨pivot downward, maintaining feasibility⟩⟩
      if every feasible neighbor of x is higher than x
          return UNBOUNDED
      else
          x ← any feasible neighbor of x that is lower than x
  return x
```

$\leftarrow \Pi$





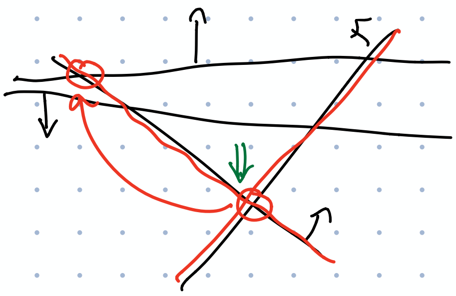
Two bases are neighbors if they share $d-1$ constraints

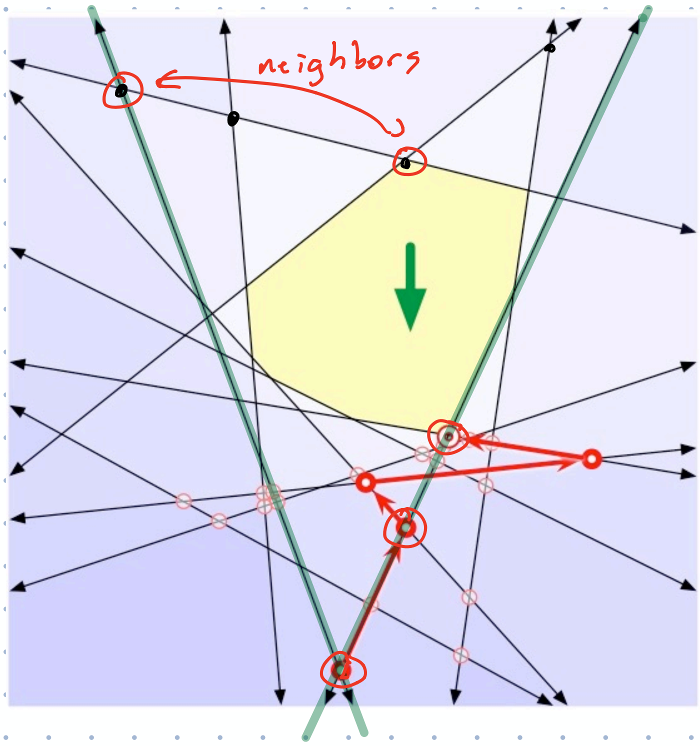
Feasibility — depends on b but not c

local optimality — depends on c but not b

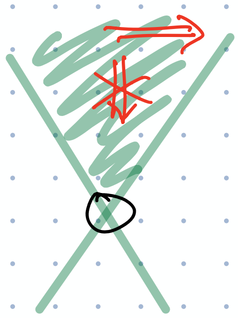
DUALSIMPLEX(H):

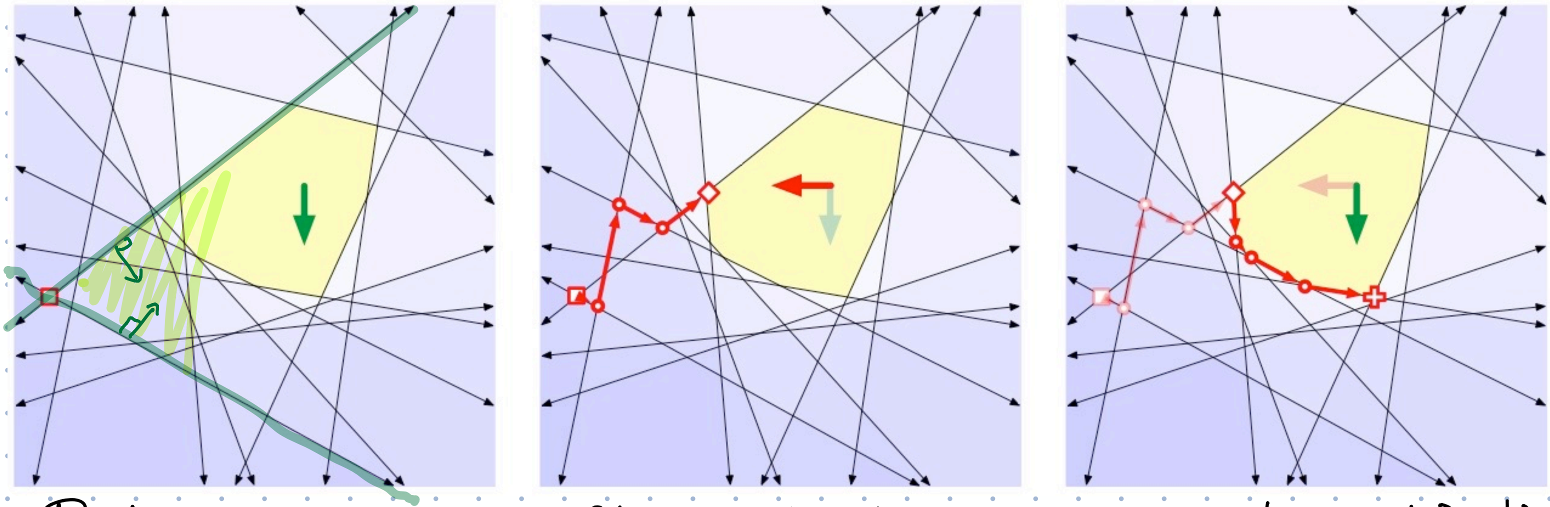
if there is no locally optimal vertex
    return UNBOUNDED
$x \leftarrow$ any locally optimal vertex / basis
while x is not feasbile
    ⟨⟨*pivot upward, maintaining local optimality*⟩⟩
    if every locally optimal neighbor of x is lower than x
        return INFEASIBLE
    else
        $x \leftarrow$ any locally-optimal neighbor of x that is higher than x
return x





Two bases are neighbors iff they share d-1 constraints





Pick any ~~basis.~~ vertex x

Change objective so that x is loc. opt.

Pivot to feasible vertex x' (dual simplex)

restore objective x' still Feasible

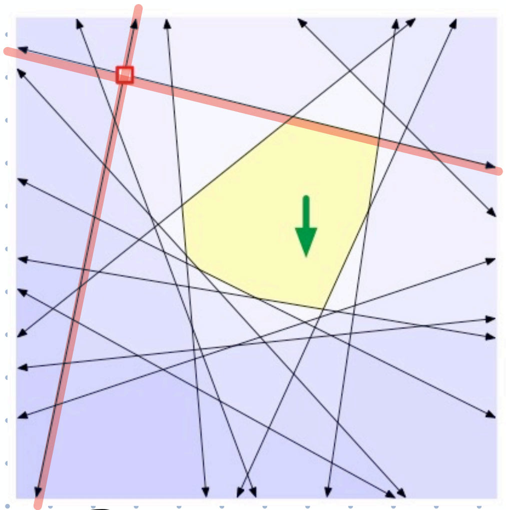
pivot to loc. opt x^*

DualPrimalSimplex(H):

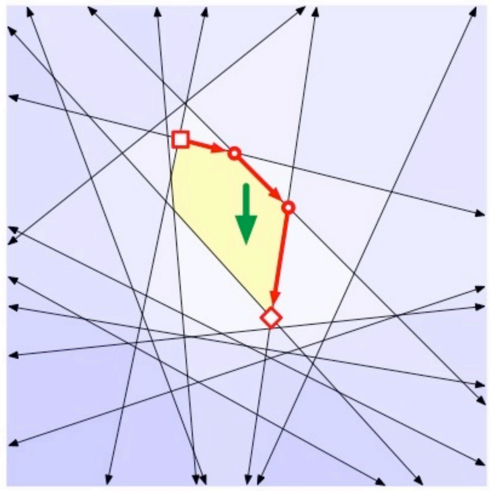
  $x \leftarrow$ any vertex
  $\tilde{H} \leftarrow$ any rotation of H that makes x locally optimal } Fast

dual
  while x is not feasible
    if every locally optimal neighbor of x is lower (wrt $\tilde{H}$) than x
      return Infeasible
    else
      $x \leftarrow$ any locally optimal neighbor of x that is higher (wrt $\tilde{H}$) than x

primal
  while x is not locally optimal
    if every feasible neighbor of x is higher than x
      return Unbounded
    else
      $x \leftarrow$ any feasible neighbor of x that is lower than x
  return x

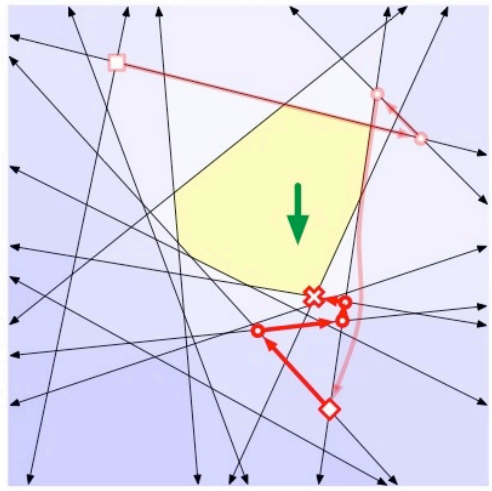


Pick any vertex x



Change offsets b so that x is feasible

Pivot down to a loc.opt ~~bas~~ vertex x'



restore offsets

Pivot up to a feasible vertex x^*

PrimalDualSimplex(H):

$x \leftarrow$ any vertex
$\tilde{H} \leftarrow$ any translation of H that makes x feasible

easy

primal

while x is not locally optimal
  if every feasible neighbor of x is higher (wrt $\tilde{H}$) than x
    return Unbounded
  else
    $x \leftarrow$ any feasible neighbor of x that is lower (wrt $\tilde{H}$) than x

dual

while x is not feasible
  if every locally optimal neighbor of x is lower than x
    return Infeasible
  else
    $x \leftarrow$ any locally-optimal neighbor of x that is higher than x
return x

Brute force enumeration: $\binom{n+d}{d} = \binom{n+d}{n}$ bases

$O(n^d)$ exponential ☹

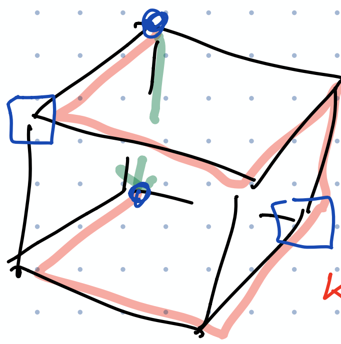
Worst case: $n=d$ $\binom{2n}{n}$ bases

$\approx \Theta(4^n/\sqrt{n})$

Feasible bases: $\Theta(n^{\lfloor d/2 \rfloor})$

$O(n)$ when $d=2$ or 3
$O(n^2)$ 4 or 5
⋮

worst case $n=d$



2d facets in d dim

2^d bases

Klee-Minty cubes

Stupid pivots → exptime

Smart pivots →

best known subexponential superpolynomial #pivots $2^{O(\log^2 d)}$

poly time?

Ellipsoid

BIG OPEN QUESTION: Pivot rule ⇒ poly #pivots?

Mostly fast

Random LP → simplex fast on average

Arbitrary LP + noise → fast on average

"smoothed analysis" $\cancel{\Theta(n + \log^2 d / \log \sigma)}$

Big open question — graph diameter of feasible polytope

Hirsch conjecture: $\leq n$ NOPE

Weak ———— $O(n)$ OPEN