Linear Programming

"programming" = table filling not coding

variables $x_1 \dots x_d \in \mathbb{R}$

objective function

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \le b_i \quad \text{for each } i = 1..p$$

$$\sum_{j=1}^{d} a_{ij} x_j = b_i \quad \text{for each } i = p+1..p+q$$

$$\sum_{j=1}^{d} a_{ij} x_j \ge b_i \quad \text{for each } i = p+q+1..n$$

constraints

mechanical

Input
- constraint matrix $A = (a_{ij}) \in \mathbb{R}^{n \times d}$
- offset vector $b \in \mathbb{R}^n$
- objective vector $c \in \mathbb{R}^d$

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \le b_i \quad \text{for each } i = 1..n$$

$$x_j \ge 0 \quad \text{for each } j = 1..d$$

"Canonical form"

↳ "standard form"

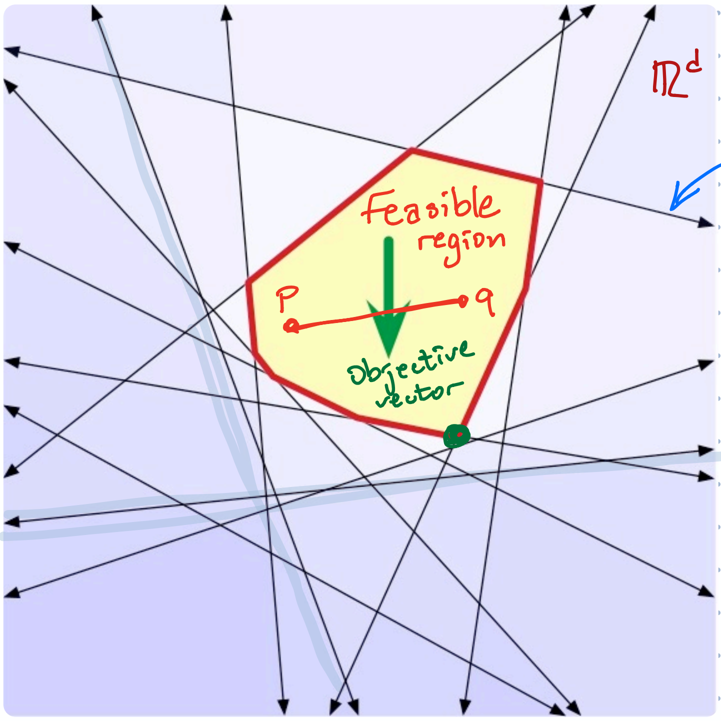
$$\boxed{\begin{aligned} \max \quad & c \cdot x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}}$$

"standard form"

Slack form

$$\boxed{\begin{aligned} \max \quad & c \cdot x \\ \text{s.t.} \quad & Ax = b \\ & x \ge 0 \end{aligned}}$$

Geometry of LP



$$\sum_{j=1}^{d} a_{ij} x_j \leq b_i$$

hyperplane (line when $d=2$
plane when $d=3$)

x is feasible iff x satisfies all the constraints

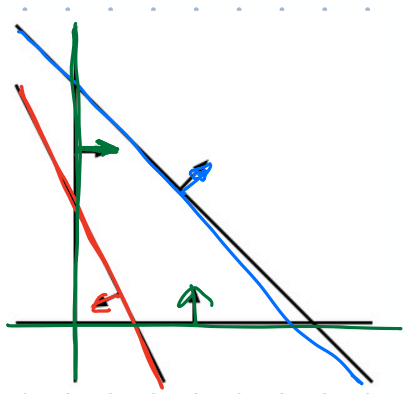
point x is feasible $\Longleftrightarrow$ x lies inside every constraint halfspace

Feasible region is a convex polyhedron

WLOG - objective = "down"

LP = Find the lowest point in a polyhedron

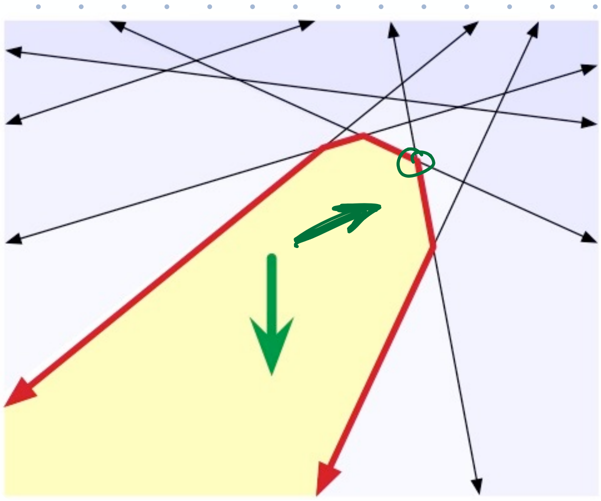
maximize	$x - y$
subject to	$2x + y \leq 1$
	$x + y \geq 2$
	$x, y \geq 0$



Infeasible

Does not depend on c

Depends on b



Unbounded

Depends on c

Does not depend on b

Shortest path

Input: Directed graph $G=(V,E)$
lengths $\ell(e)$ for each $e \in E$
two vertices s,t.

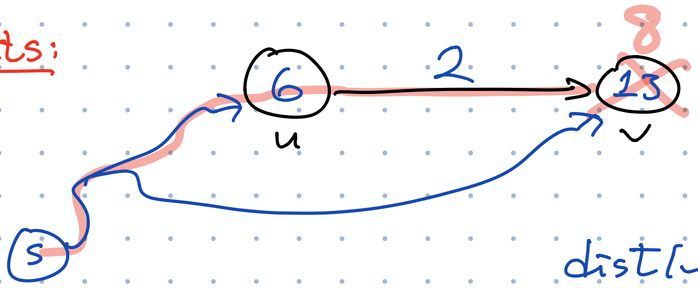
Output: length of shortest walk from s to t in G

maximize	$dist(t)$	
subject to	$dist(s) = 0$	
	$dist(v) - dist(u) \le \ell(u \to v)$	for every edge $u \to v$

No edge is tense

Variables: $dist(v)$ — distance from s to v

Constraints:



[Ford '53]

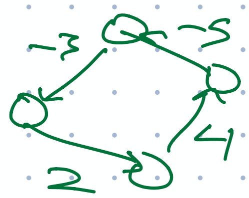
$u \to v$ is tense if

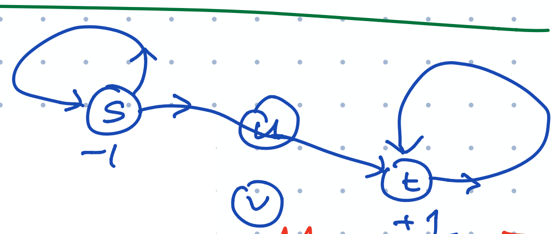
$dist(u) + \ell(u \to v) < dist(v)$

When no edge is tense, dist correct

Feasible $\leftrightarrow$ no neg cycles reachable from s

Unbounded $\leftrightarrow$ no path from s to t





minimize	$\sum_{u \to v} \ell(u \to v) \cdot x(u \to v)$	
subject to	$\sum_u x(u \to t) - \sum_w x(t \to w) = 1$	
	$\sum_u x(u \to v) - \sum_w x(v \to w) = 0$	for every vertex $v \ne s, t$
	$x(u \to v) \ge 0$	for every edge $u \to v$

Min-cost Flow
$x(u \to v)$ = flow value
$\ell(u \to v)$ = cost
$b(s) = -1$ $b(t) = +1$

Variables: $x(u \to v)$ for each edge $u \to v$

1 if $pred(v) = u$ — $u \to v$ on shortest path $s \to t$
0 otherwise

Constraints: walk from s to t

must enter t once more than exits t

For any vertex except s,t, #enter = #exit

walk traverse each edge ≥ 0 times

Objective: sum of lengths of edges

Infeasible: no walk from s to t
Unbounded: negative cycle

Example #2: Maximum flow

$$\begin{aligned}
\text{maximize} \quad & \sum_w f(s\to w) - \sum_u f(u\to s) & & \\
\text{subject to} \quad & \sum_w f(v\to w) - \sum_u f(u\to v) = 0 & & \text{for every vertex } v \neq s,t \\
& f(u\to v) \le c(u\to v) & & \text{for every edge } u\to v \\
& f(u\to v) \ge 0 & & \text{for every edge } u\to v
\end{aligned}$$

balance

capacity

non-neg

Variables: $f(u\to v)$ flow values $f \in \mathbb{R}^E$

Constraints: balance, capacity, non-negativity

Objective: net flow out of s

Infeasible — impossible! 0 is feasible (if $c \ge 0$)

Unbounded — impossible! (if $c \le \infty$)

$|f^*| \le \sum_v c(s\to v)$

Minimum cut:

$$\begin{aligned}
\text{minimize} \quad & \sum_{u\to v} c(u\to v)\cdot x(u\to v) & & \\
\text{subject to} \quad & x(u\to v) + S(v) - S(u) \ge 0 & & \text{for every edge } u\to v \\
& x(u\to v) \ge 0 & & \text{for every edge } u\to v \\
& S(s) = 1 & & \\
& S(t) = 0 & &
\end{aligned}$$

Variables: "$S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \in T \end{cases}$"

"$x(u\to v) = \begin{cases} 1 & \text{if } u \in S \text{ and } v \in T \\ 0 & \text{otherwise} \end{cases}$"

Objective: capacity $\|S,T\| = \sum_{u\in S}\sum_{v\in T} c(u\to v)$

Duality

$c \to$ variables
$b \to$ constraints

Primal (Π)

$$\begin{aligned} \max \quad & c \cdot x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

$\Longleftrightarrow$

Dual (Π)

$$\begin{aligned} \min \quad & y \cdot b \\ \text{s.t.} \quad & yA \ge c \\ & y \ge 0 \end{aligned}$$

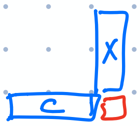
$b \to$ varibles
$c \to$ constraints

Dual (Π)

$$\begin{aligned} \max \quad & -b^\top \cdot y^\top \\ \text{s.t.} \quad & -A^\top y^\top \le -c^\top \\ & y^\top \ge 0 \end{aligned}$$

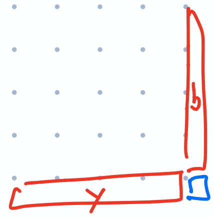
canonical form

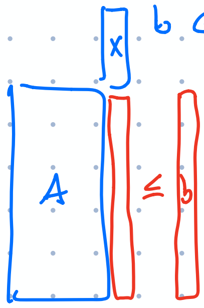
canonical form

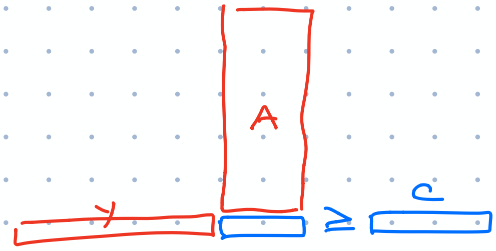


$c \in \mathbb{R}^d$
$x \in \mathbb{R}^d$
$A \in \mathbb{R}^{n \times d}$
$b \in \mathbb{R}^n$

$b \in \mathbb{R}^n$
$y \in \mathbb{R}^n$
$A \in \mathbb{R}^{n \times d}$
$c \in \mathbb{R}^d$







Dualizing general LPs:

Primal	Dual	Primal	Dual
$\max c \cdot x$	$\min y \cdot b$	$\min c \cdot x$	$\max y \cdot b$
$\sum_j a_{ij} x_j \le b_i$	$y_i \ge 0$	$\sum_j a_{ij} x_j \le b_i$	$y_i \le 0$
$\sum_j a_{ij} x_j \ge b_i$	$y_i \le 0$	$\sum_j a_{ij} x_j \ge b_i$	$y_i \ge 0$
$\sum_j a_{ij} x_j = b_i$	–	$\sum_j a_{ij} x_j = b_i$	–
$x_j \ge 0$	$\sum_i y_i a_{ij} \ge c_j$	$x_j \le 0$	$\sum_i y_i a_{ij} \ge c_j$
$x_j \le 0$	$\sum_i y_i a_{ij} \le c_j$	$x_j \ge 0$	$\sum_i y_i a_{ij} \le c_j$
–	$\sum_i y_i a_{ij} = c_j$	–	$\sum_i y_i a_{ij} = c_j$
$x_j = 0$	–	$x_j = 0$	–

Shortest paths:

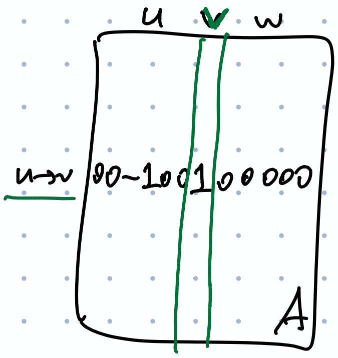
$d = V$
$n = E$

$$
\begin{array}{lll}
\text{maximize} & dist(t) & \\
\text{subject to} & dist(s) = 0 & \\
& dist(v) - dist(u) \le \ell(u \to v) & \text{for every edge } u \to v
\end{array}
$$

Variables: $x = dist \in \mathbb{R}^V$

Objective vector $= (0, 0, 0, \ldots 0, \overset{t}{1}) = c$

Offset vector $= \ell \in \mathbb{R}^E = b$



Constraint matrix $A \in \mathbb{R}^{n \times d} = \mathbb{R}^{E \times V}$

$$
A[u \to v, w] = \begin{cases} +1 & \text{if } w = v \\ -1 & \text{if } w = u \\ 0 & \text{otherwise} \end{cases}
$$

signed incidence matrix

Dual LP variables: $y(u \to v)$ for every edge $u \to v$

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{u \to v} y(u \to v) \cdot \ell(u \to v) & \\
\text{such that} & y(u \to v) \ge 0 & \text{for all } u \to v \\
& \displaystyle\sum_u y(u \to v) - \sum_w y(v \to w) = 0 & \text{for all } v \ne s, t \\
& \displaystyle\sum_u y(u \to t) - \sum_w y(t \to w) = 1 &
\end{array}
$$

The Fundamental Theorem of Linear Programming. *A canonical linear program* Π *has an optimal solution* x^* *if and only if the dual linear program* $\amalg$ *has an optimal solution* y^* *such that* $c \cdot x^* = y^*Ax^* = y^* \cdot b$.