Standard Flow network:

Changes

Directed graph $G=(V,E)$

Capacity $c(e)\geq 0$ for each $e\in E$

~~Two vertices $s,t\in V$~~

Balance $b(v)$ for every vertex v s.t. $\sum_v b(v)=0$

Feasible flow =

$f: E\to\mathbb{R}$ $\qquad 0\leq f(e)\leq c(e)$

$\sum_w f(v\to w)-\sum_u f(u\to v)=\cancel{0}\ b(v)$ for all vertices v ~~$\neq s,t$~~

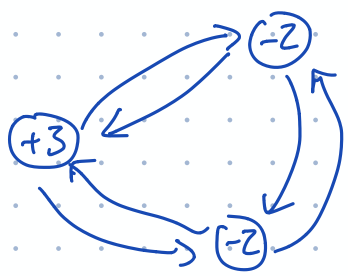
~~Maximize~~

$\cancel{|f|=\sum_w f(s\to w)-\sum_u f(u\to s)}$

Find a feasible flow if one exists

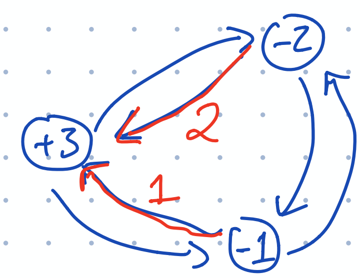
$b(v)>0\Rightarrow$ demand consumption

$b(v)<0\Rightarrow$ supply production



no feasible flow

because $\sum_v b(v)\neq 0$

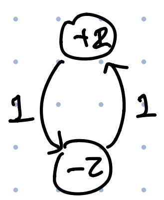


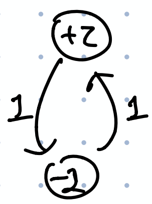
Feasible

f is feasible $\Rightarrow \sum_v b(v)=\sum_v\left(\sum_u f(u\to v)-\sum_w f(v\to w)\right)$

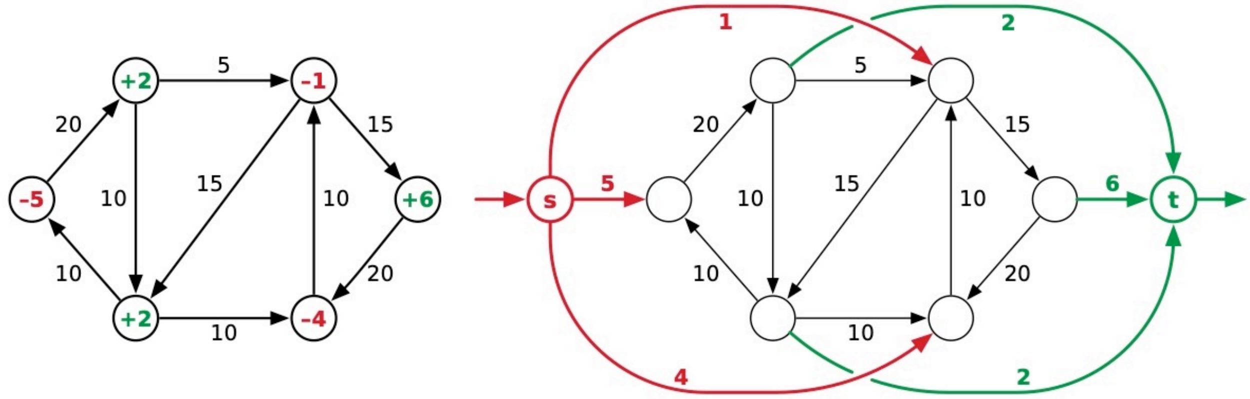
$$=\sum_{u\to v} f(u\to v)-\sum_{v\to w} f(v\to w)=0$$

$\sum_v b(v)=0$ is necessary but not sufficient





Solve by reducing to standard max Flow



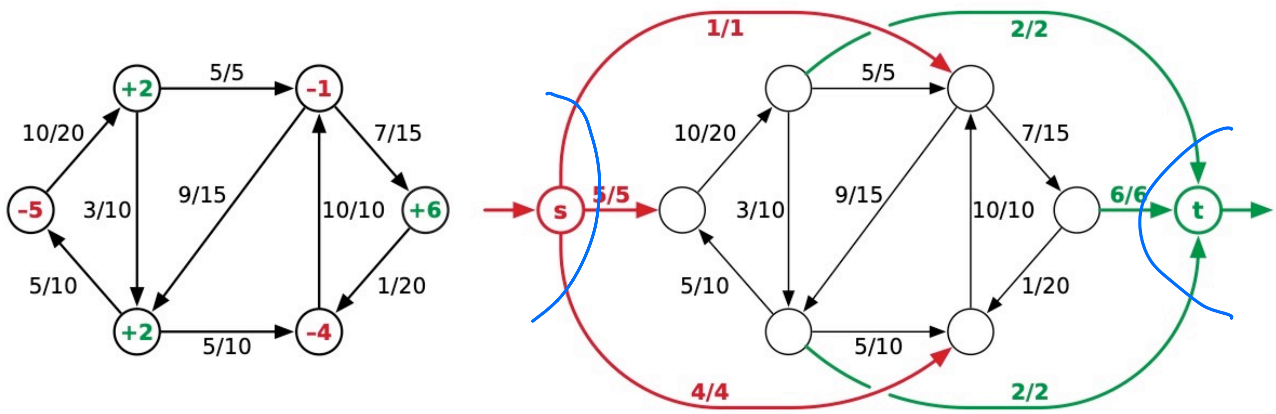
A flow network G with non-zero balance constraints, and the transformed network G'.

Given $G=(V,E)$
$c(e)$ for all $e \in E$
$b(v)$ for all $v \in V$
$\longrightarrow$
Construct $G'=(V', E')$

$V' = V \cup \{s,t\}$

$E' = E \cup \{s \to v \mid b(v) < 0\}$
$\cup \{v \to t \mid b(v) > 0\}$

$c(s \to v) = -b(v)$
$c(v \to t) = b(v)$



A feasible flow in G and the corresponding saturating flow in G'.

Feasible flow f in G $\Longleftrightarrow$ Saturating flow f' in G'

↳ Max Flow via Orlin's $O(V'E') = O(VE)$ time

$\Rightarrow$ Let f be any feasible flow in G

$$\text{Define } f'(u \to v) = \begin{cases} f(u \to v) & \text{if } u \to v \in E \\ -b(v) & \text{if } u = s \\ b(u) & \text{if } v = t \end{cases} \implies f' \text{ is saturating}$$

$\Leftarrow$ f' saturating $\Longrightarrow$ $f'|_E = f$ is feasible

Max-Flow min-cut theorem?

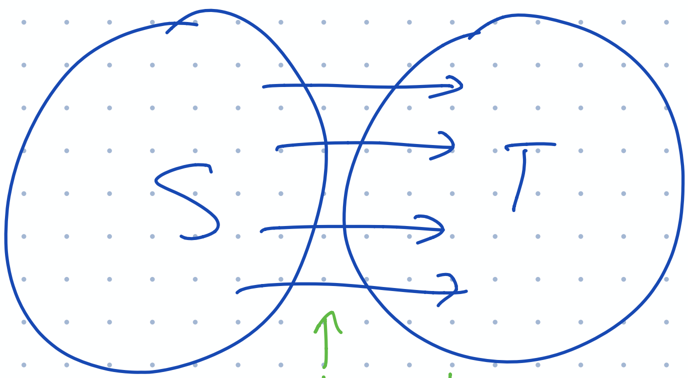
For any x
Either $|f| \geq x$
or $\|S,T\| \leq x$

In a network with non-zero $b(v)$

a cut (S,T) is infeasible if

$$\|S,T\| = \sum_{u \in S} \sum_{v \in T} c(u \to v) < \underbrace{\sum_{v \in T} b(v)}_{\text{demand of } T} = b(T)$$

wants to produce $-b(S)$



wants to consume $b(T)$.

most can be transmitted is $\|S,T\|$

Every flow network has either feasible flow or infeasible cut

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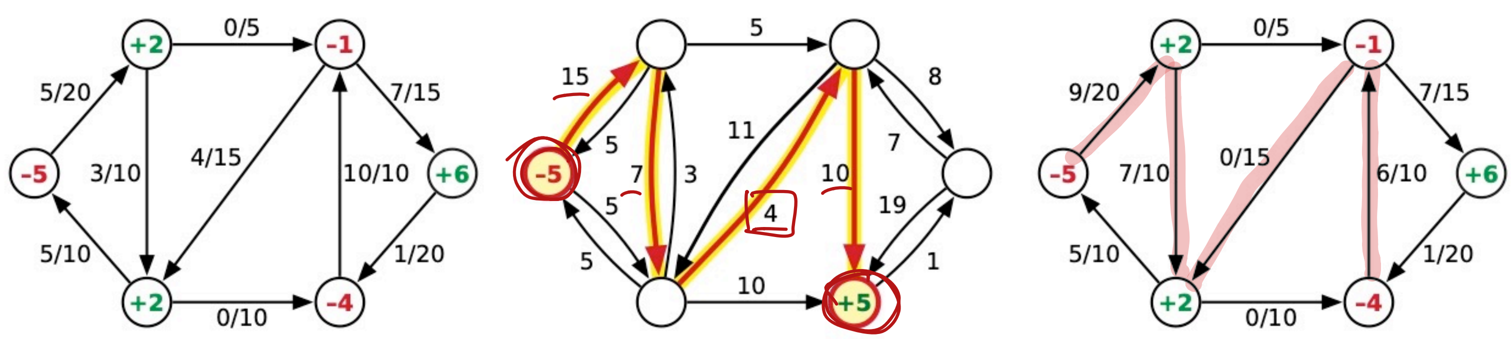
Ford-Fulkerson?
Augmenting path?
Residual graph?

pseudoflow: $f: E \to \mathbb{R}$

Feasible: $0 \leq f(e) \leq c(e)$ for all e
balanced: $\sum_u f(u \to v) - \sum_w f(v \to w) = b(v)$ for all v

flow = balanced pseudoflow



From left to right: A pseudoflow ψ in a flow network G; the residual graph G_ψ with one augmenting path highlighted; and the updated pseudoflow after pushing 4 units along the augmenting path.

Aug path = path from u to v in G_f where $b_f(u) < 0$ and $b_f(v) > 0$

$$b_f(v) = b(v) - \left(\sum_u f(u \to v) - \sum_w f(v \to w)\right)$$

FF

```
FeasibleFlow(V, E, c, b):
  for every edge e ∈ E
      ψ(e) ← 0
  B ← ∑_v |b(v)|/2
  while B > 0
      construct G_ψ
      ⟨⟨Find augmenting path π⟩⟩
      s ← any vertex with b_ψ(s) < 0
      if s cannot reach a vertex t in G_ψ with b_ψ(t) > 0
          return Infeasible
      t ← any vertex reachable from s with b_ψ(t) > 0
      π ← any path in G_ψ from s to t
      ⟨⟨Push as much flow as possible along π⟩⟩
      R ← min{−b_ψ(s), b_ψ(t), min_{e∈π} c_ψ(e)}
      B ← B − R
      for every directed edge e ∈ π
          if e ∈ E
              ψ(e) ← ψ(e) + R
          else ⟨⟨rev(e) ∈ E⟩⟩
              ψ(e) ← ψ(e) − R
  return ψ
```

Integer $\longrightarrow$ $O(EB)$ time

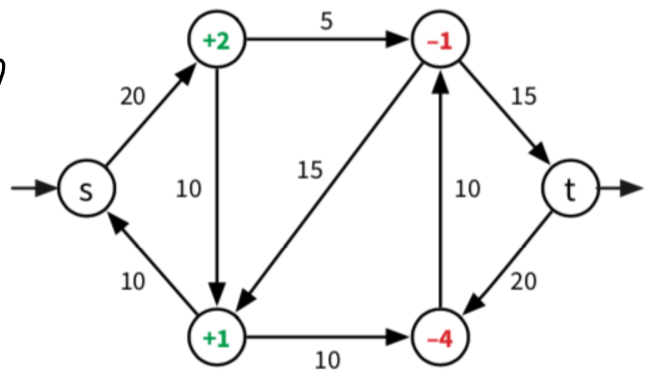
Orlin's $\longrightarrow$ $O(EV)$ time

Maximum flow with non-zero balances

1. Feasible (balanced) flow f in G ?
2. Find max flow f' in G_f ← standard
3. Return $f+f'$

value $= |f| + |f'|$

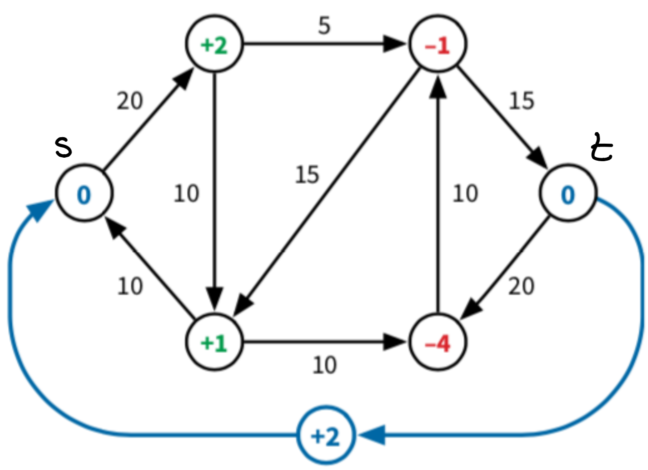
G



feasible flow f

might have non zero value

G'

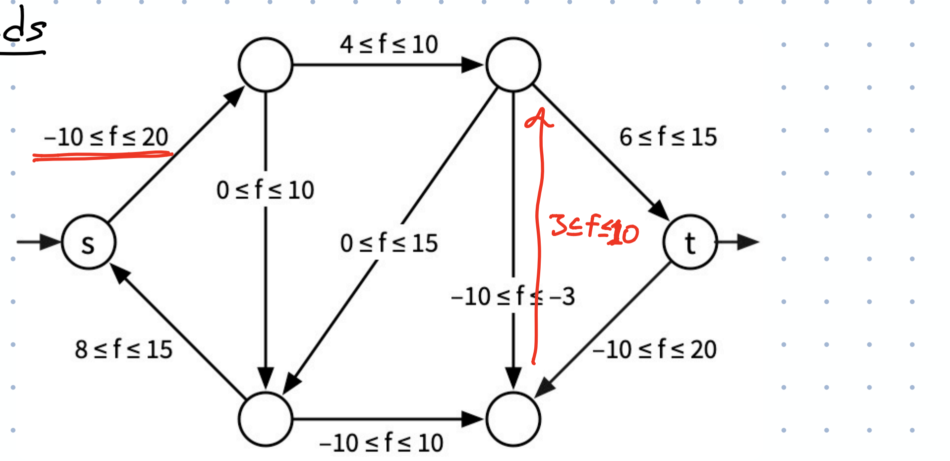


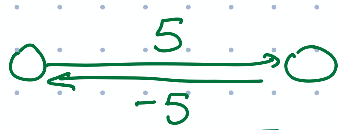
Feasible (balanced) flow

O(VE) time

Max-Flow with lower bounds on the edges

1. Find feasible flow f in G
2. Find max flow f' in G_f
3. Return $f+f'$

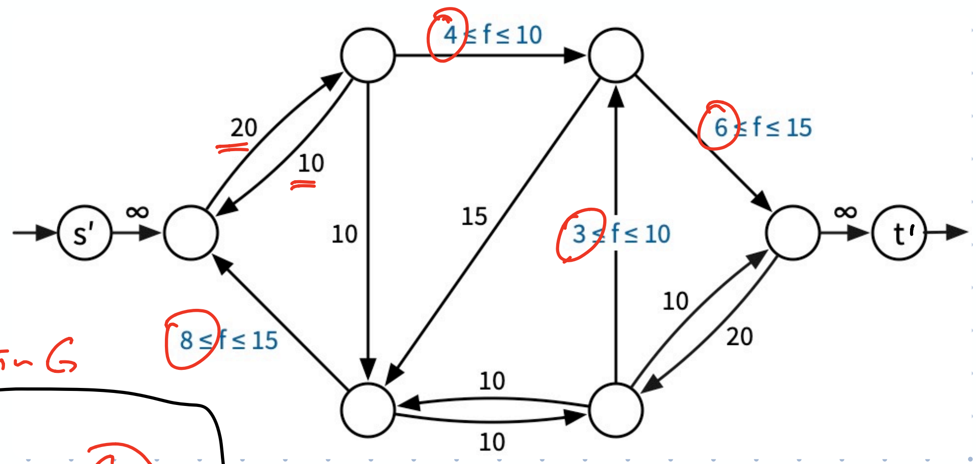




$f(u \to v) = -f(v \to u)$

$c(u \to v) := -\ell(v \to u)$

$\ell(u \to v) := -c(v \to u)$

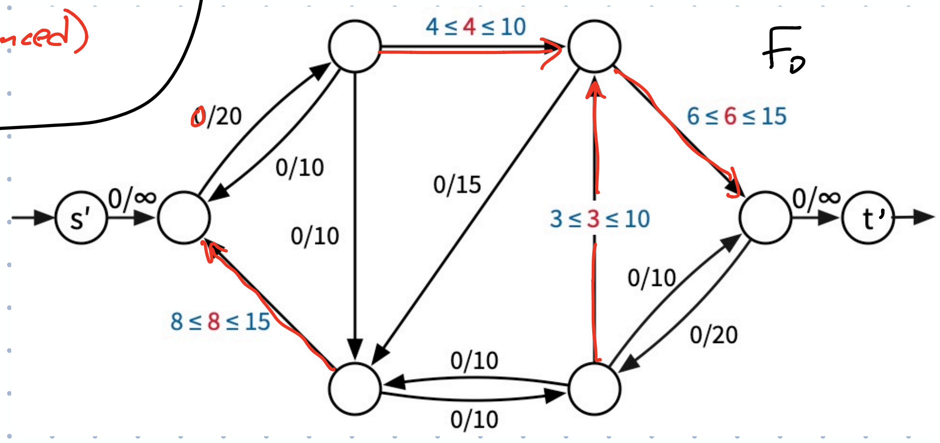


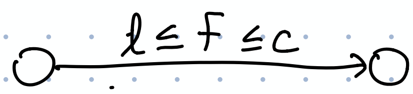
1(a) Find a feasible pseudoflow f_0 in G

(b) Find a feasible balanced flow f_1 in G_{f_0}

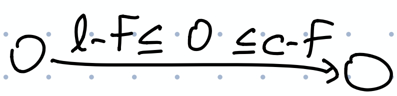
2 Find max(feasible, balanced) flow in G_{f_1}

= Max flow in G_{f_0}

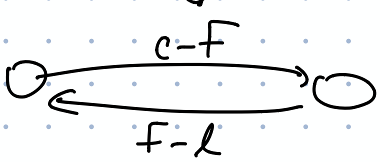


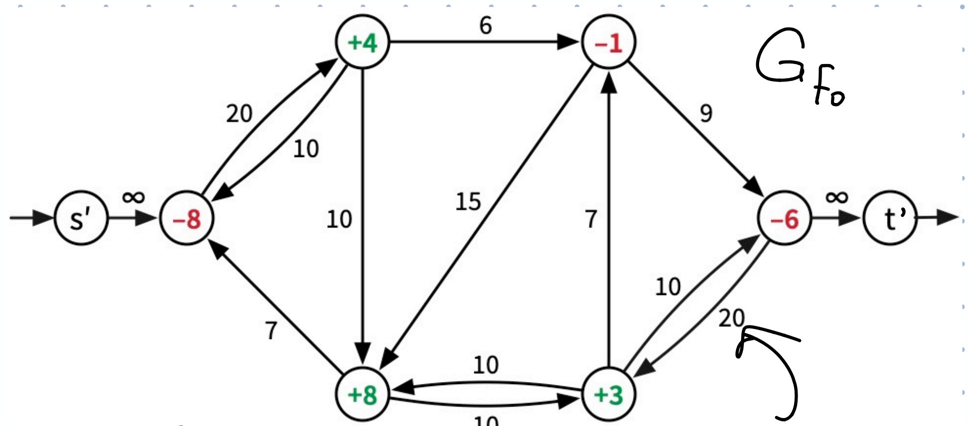


$\Downarrow$



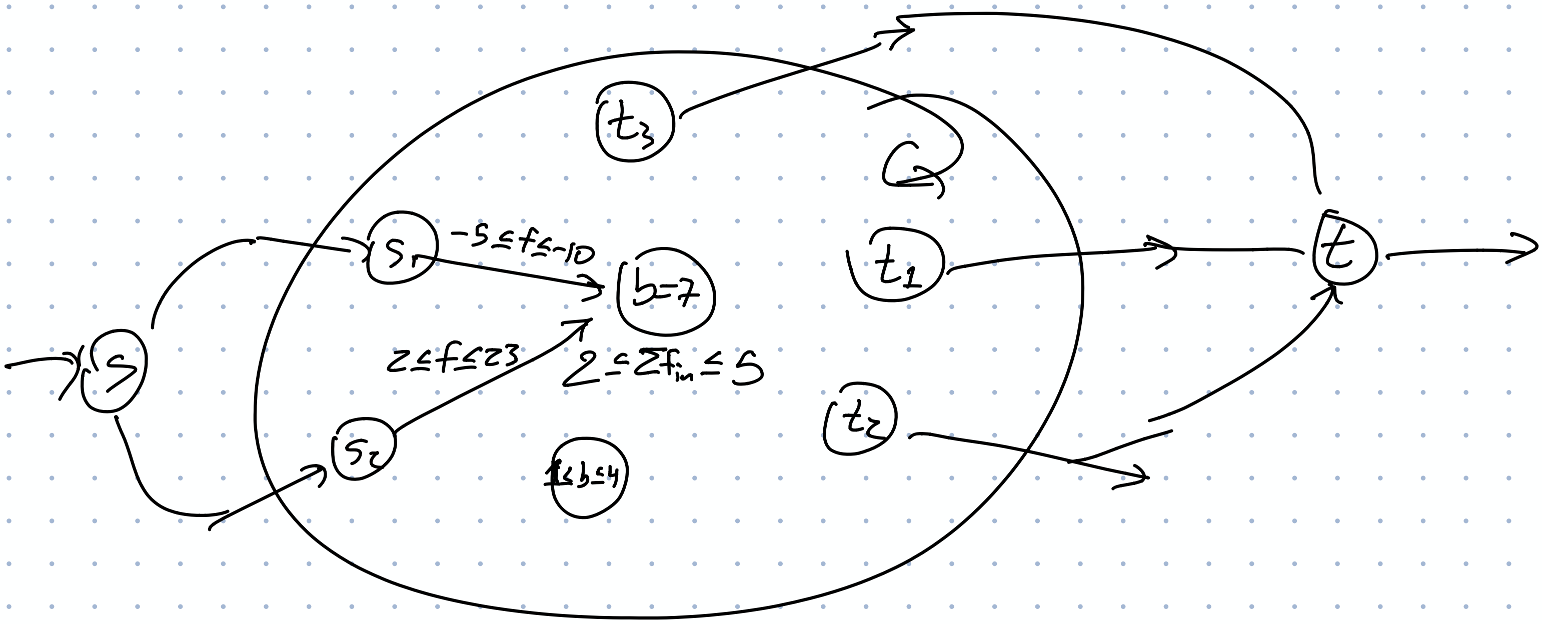
$\Downarrow$





$\boxed{O(VE)\text{ time}}$

Find a max flow in this



s
s_1
s_2
$-5 \le f \le -10$
$2 \le f \le 23$
$b=7$
$2 \le \Sigma f_{in} \le 5$
$1 \le b \le 4$
t_3
t_1
t_2
t