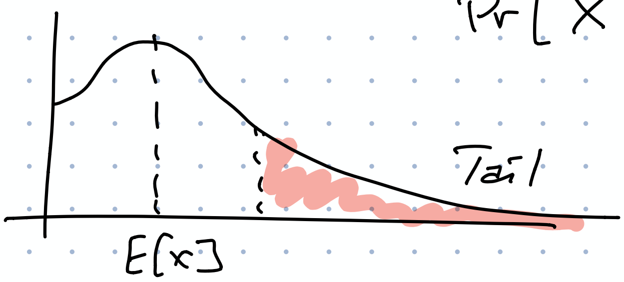
Quicksort runs in $O(n \log n)$ time with high probability

depends on c with prob. $\leq 1 - \frac{1}{n^c}$

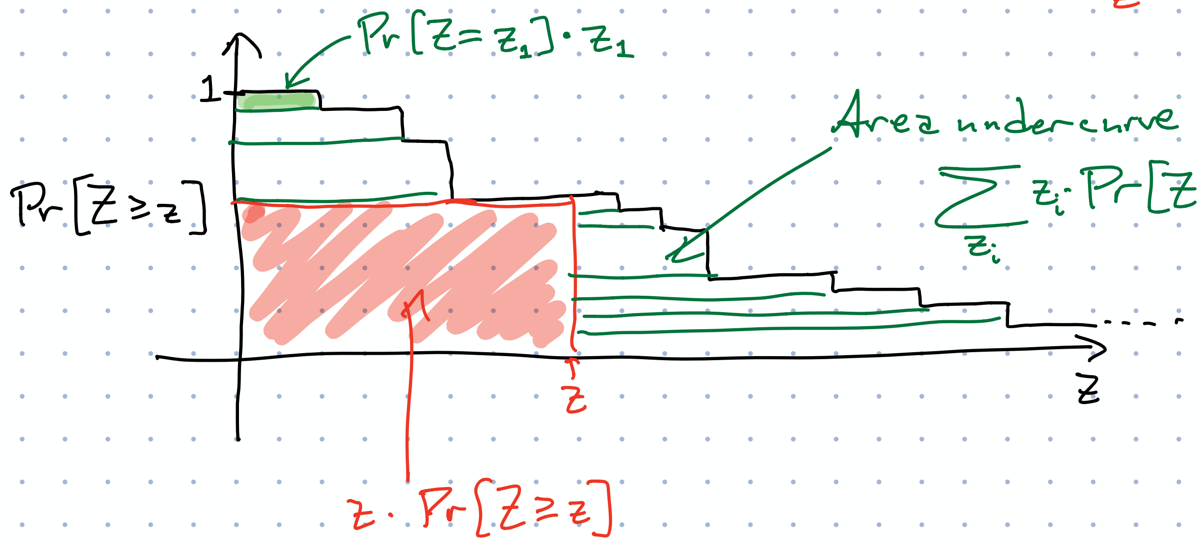
Tail inequality

$$\Pr[X \geq (1+\delta) \cdot E[X]] \leq \_\_\_\_$$



Markov's Inequality: If Z is any non-neg integer random var

$$\Pr[Z \geq z] \leq \frac{E[Z]}{z} = \frac{\mu}{z}$$



Area under curve

$$\sum_{z_i} z_i \cdot \Pr[Z = z_i] = E[Z]$$

$$\Pr[X \geq (1+\delta) E[X]] \leq \frac{1}{1+\delta}$$

$$\Pr[\text{Quicksort runs in } O(n^3) \text{ time}] \leq 1 - O\left(\frac{\log n}{n^2}\right)$$

X and Y are independent

iff $\Pr[X = x \text{ and } Y = y] = \Pr[X = x] \cdot \Pr[Y = y]$

iff $\Pr[X = x \mid Y = y] = \Pr[X = x]$

$\Rightarrow E[X \cdot Y] = E[X] \cdot E[Y]$ $\quad \Rightarrow f(X)$ and $f(Y)$ are independent

$X_1 \dots X_n$ are fully indep.

$$\Pr\left[\bigwedge_{i=1}^{n} (X_i = x_i)\right] = \prod_{i=1}^{n} \Pr(X_i = x_i)$$

$$\Rightarrow \quad E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$$

$X_1 \dots X_n$ are k-wise indep. if every subset of size k is fully indep.

Coins X, Y, Z $H=1$ $T=0$ $W = X+Y+Z \bmod 2$

$\{W, X, Y, Z\}$ is 3-way indep
not 4-way indep

$k=2 \Rightarrow$ pairwise independent

$$X = \sum_{i=1}^{n} X_i \qquad X_i \in \{0,1\}$$

$$p_i = \Pr[X_i = 1] = E[X_i]$$

$$\mu = E[X] = \sum_i E[X_i] = \sum_i p_i$$

Chebyshev's Inequality:

If $X_1 \dots X_n$ are pairwise independent

then $\boxed{\Pr\left[(X-\mu)^2 \ge z\right] \le \frac{\mu}{z}}$.

$\mathrm{Var}[X] = E[(X-\mu)^2]$

Proof:

Define $Y_i = X_i - p_i$ $\quad Y = X - \mu$

$$E[Y^2] = E\left[\sum_{i,j} Y_i Y_j\right] = \sum_{i,j} E[Y_i Y_j]$$

$$= \sum_i E[Y_i^2] + \sum_{i \ne j} E[Y_i \cdot Y_j]$$

$$= \sum_i E[Y_i^2] + \cancel{\sum_{i \ne j} \underset{0}{E[Y_i]}\, \underset{0}{E[Y_j]}}$$

$$= \sum_i \Big(p_i \underbrace{(1-p_i)^2}_{Y_i^2 \text{ when } X_i=1} + (1-p_i)p_i^2\Big) \leq \mu$$

$Pr[X_i=1]$

$E[Y^2] \leq \mu$

$Pr[Y^2 \geq z] \leq \frac{\mu}{z}$ by Markov $\square$

$$Pr[X \geq (1+\delta)\mu] \leq \frac{1}{\delta^2 \mu}$$

$$Pr[X \leq (1-\delta)\mu] \leq \frac{1}{\delta^2 \mu}$$

$2k^{th}$ Moment Ineq.

If $X_1 .. X_n$ are $2k$-indep. then

$$Pr[(X-\mu)^{2k} \geq z] \leq O(\mu^k / z)$$

$$\Rightarrow Pr[X \geq (1+\delta)\mu] \leq O\left(\left(\frac{1}{\delta^2 \mu}\right)^k\right)$$

Exponential Moment Ineq. $X_1 .. X_n$ are fully indep.

then $E[\alpha^X] \leq e^{(\alpha-1)\mu}$ for any $\alpha \geq 1$

Proof: $E[\alpha^{X_i}] = p_i \cdot \alpha^1 + (1-p_i)\alpha^0 = (\alpha-1)p_i + 1$

TWMUF: $\boxed{1+t \leq e^t}$ $\leq e^{(\alpha-1)p_i}$

$$E[\alpha^X] = \prod_i E[\alpha^{X_i}] \leq \prod_i e^{(\alpha-1)p_i} = e^{(\alpha-1)\mu} \quad \square$$

$Pr[X \geq x] \leq e^{x-\mu} (\mu/x)^x$ for all $x \geq \mu$.

Proof: Let $\alpha = \frac{x}{\mu}$ Do math.

Chernoff bound

$$Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{\mu} \leq e^{-\delta^2 \mu / 3}$$

Treaps

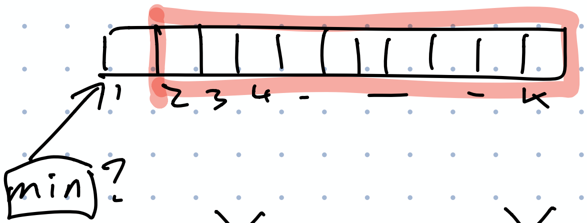
$E[\text{depth}(k)] = \sum_i \Pr[i \uparrow k]$

$X = \text{depth}(k)$ $\qquad X_i = [i \uparrow k]$

Claim: $[1\uparrow k], [2 \uparrow k], \ldots, [k-1 \uparrow k]$ are fully indep.

$\downarrow$

$\text{priority}(1) = \min\{\text{priority}(j) \mid 1 \le j \le k\}$



$$\text{depth}(k) = \underbrace{\overbrace{\sum_{i<k} [i \uparrow k]}^{X}}_{\substack{E[''] = H_{k+1} - 1 \\ < \ln n}} + \underbrace{\overbrace{\sum_{i>k} [i \uparrow k]}^{Y}}_{\substack{H_{n-k} - 1 \\ < \ln n}}$$

$$\Pr[X \ge 4 \ln n] \le \left(\frac{e^3}{4^4}\right)^{\ln n} \le \frac{1}{n^{2.5\text{ish}}}$$

$$\Pr[Y \ge 4 \ln n] \le \frac{1}{n^{2.5\text{ish}}}$$

$$\Pr[\text{depth}(k) \ge 8 \ln n] \le \frac{2}{n^{2.5\text{ish}}}$$

$$\Pr[\max_k \text{depth} \ge 8 \ln n] \le n \cdot \frac{2}{n^{2.5\text{ish}}} = \frac{2}{n^{1.5\text{ish}}}$$

$$\Pr[X \ge 4 \ln n] = \Pr\left[X \ge \underbrace{\left(4 \frac{\ln n}{\ln k}\right)}_{1+\delta} \ln k\right] \le \left(\frac{e^{4\frac{\ln n}{\ln k} - 1}}{(4 \log_k n)^{\log_k n}}\right)^{\ln k}$$