Randomized Algorithms

$$\underset{|x|=n}{E}[T(x)] = O(n \log n)$$

Unreasonable to assume input distribution

$$\max_{|x|=n} E[T(x)] = O(n \log n)$$

randomness in the algorithm

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sample space = finite/countable set Ω

Prob. mass Function $Pr : \Omega \to \mathbb{R}$

$$Pr(\omega) \geq 0 \qquad \sum_{\omega \in \Omega} Pr(\omega) = 1$$

$\Omega = \{\text{rock}, \text{paper}, \text{scissors}\}$ $Pr(\text{rock}) = 1$ $Pr[p] = 0$ $pr[s] = 0$

Event = subset of Ω = condition/proposition

$$Pr[A] = \sum_{\omega \in A} Pr(\omega)$$

$A \vee B$ $\quad A \wedge B$ $\quad \neg A$

red die + blue die

$$Pr[\text{at most one } 5] = Pr[\neg(\text{roll } 2\ 5\text{s})] = 1 - Pr[\text{roll } 2\ 5\text{'s}]$$
$$= \frac{35}{36}$$

Conditional Probability $Pr[A \mid B] = \dfrac{Pr[A \wedge B]}{Pr[B]}$

$$Pr[\text{red } 5 \mid \text{at least one } 5] = 6/11$$

$$Pr[\underset{\uparrow\ 10}{\text{a least one } 5} \mid \underset{\uparrow\ 35}{\text{at most one } 5}] = 2/7$$

A, B <u>disjoint</u> $Pr[A \wedge B] = 0$ $\qquad A \cap B = \emptyset$

A, B <u>independent</u> if $Pr[A \wedge B] = Pr[A] \cdot Pr[B]$

Random variable $X : \Omega \to V$ value set

$X : \Omega \to \mathbb{Z}$ random integer
int random variable

$\Pr\{X = x\}$ $\Pr[X \geq x]$

Expectation:

$$E[X] = \sum_x x \cdot \Pr[X = x]$$

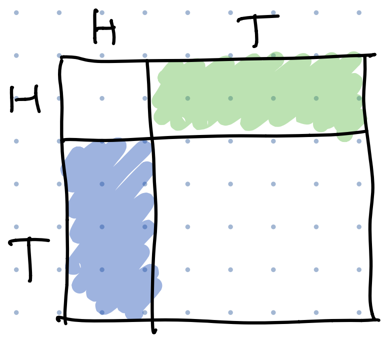
X = # top of 6-sided die $E[x] = 1\frac{1}{6} + 2\frac{1}{6} + \cdots + 6\frac{1}{6} = 3\frac{1}{2}$

John von Neumann 1945

Biased coin: $\Pr[\text{heads}] = \overset{\text{unknown}}{p}$ $\Pr[\text{tails}] = 1-p$

Flip twice. $HT \to 0$ $TH \to 1$

else "recurse"



$\Pr[0 \mid \text{Flip1} \neq \text{Flip2}] = 1/2$

$$E[\#\text{Flips}] =$$

$$\Pr[f1 \neq f2]\, \underbrace{E[\#F \mid \text{Flip1} \neq \text{Flip2}]}_{= 2} + \overset{\Pr[f1 = f2]}{} \underbrace{E[\#F \mid \text{Flip1} = \text{Flip2}]}_{= 2 + E[\#F]}$$

Condition Exp.

$$E[X \mid A] = \sum_x \Pr[X = x \mid A] \cdot x$$

$$E[\#F] = \underbrace{\Pr[f1 \neq f2]}_{2p(1-p)} \cdot 2 + \underbrace{\Pr[f1 = f2]}_{p^2 + (1-p)^2} (2 + E[\#F])$$

$$\boxed{E[\#F] = 1/p(1-p)}$$

Pokémon

Suppose there are n species

Suppose 1 card at a time
species uniformly at random

How many cards do you have to buy to catch 'em all?

$E[\#\text{cards}] = ?$

Linearity of Expectation: $E[X+Y] = E[X] + E[Y]$

$X(n) = $ # cards to get all n Pokémon

$X(i) = $ # cards to get i distinct Pokemon

$E[X] = E[Y_1] + E[Y_2] + E[Y_3] + \cdots + E[Y_n] = \sum_{i=1}^{n} \frac{n}{n-i+1}$

$Y_i = $ # cards after we have $i-1$ P. to get ith P.

= biased coin

$Pr[\text{heads}] = \frac{n-i+1}{n}$

$\overset{\shortparallel}{\text{new Pok}}$

$E[Y_i] =$

$Pr[\text{heads}] \cdot E[Y_i \mid \text{heads}] + Pr[\text{tails}] \cdot E[Y_i \mid \text{tails}]$

$= \frac{n-i+1}{n} \cdot 1 + \frac{i-1}{n}(1 + E[Y_i])$

$= \frac{n}{n-i+1}$

$E[X] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \sum_{i=1}^{n} \frac{1}{n-i+1} = n \sum_{j=1}^{n} \frac{1}{j}$ $\quad (j = n-i+1)$

$= \boxed{n H_n} = \underline{O(n \log n)}$

$\boxed{\ln(n+1) \le H_n \le \ln n + 1}$

Shuffling Cards

DrawLots(L):

```
n ← |L|
for i ← 1 to n
    remove random lot x from L
    R[i] ← x
return R[1..n]
```

uniform, indep.

Fisher-Yates

SelectionShuffle(A[1..n]):

```
for i ← n down to 1
    swap A[i] ↔ A[RANDOM(i)]
```

$O(n)$ time

uniform from $\{1, 2, \ldots, i\}$