Midterm 1 is graded (and regraded)

HW 6 is due next Tue.

Maximum Flows:

Input: $G=(V,E)$ directed
$s,t \in V$ source, target
$c: E \to \mathbb{R}^+$ capacities

Output: $f: E \to \mathbb{R}$

$$0 \le f(u\to v) \le c(u\to v)$$

$$\sum_u f(u\to v) = \sum_w f(v\to w) \quad \text{for all } v \ne s,t$$

$$\text{maximize } \sum_u f(u\to t) - \sum_w f(t\to w)$$

Ford-Fulkerson [54] augmenting path algo

$f \leftarrow 0$
construct residual graph G_f
while G_f has a path from s to t
  push flow along path
  update f
  update G_f
return f

$O(V+E)$

$$c_f(u\to v) = \begin{cases} c(u\to v) - f(u\to v) & \text{if } u\to v \in E \\ f(v\to u) & \text{if } v\to u \in E \end{cases}$$

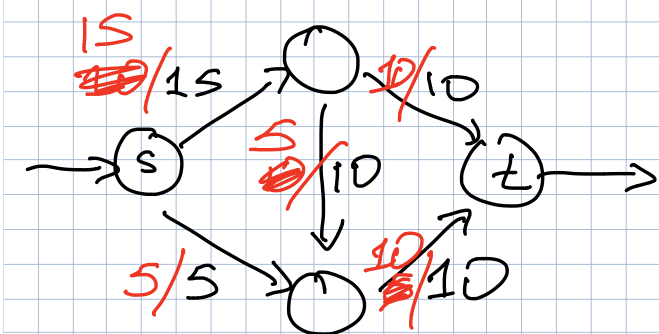
Integer capacities:

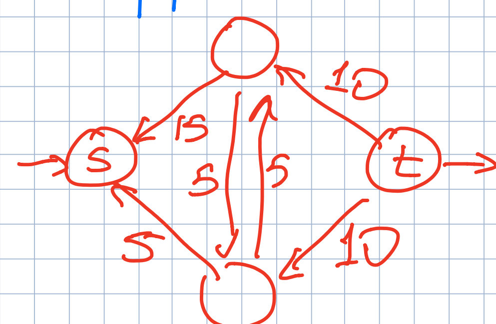
iters ≤ max flow value

exponential in worst case!

Real capacities: ∞ loops possible

don't even converge to a good approx.



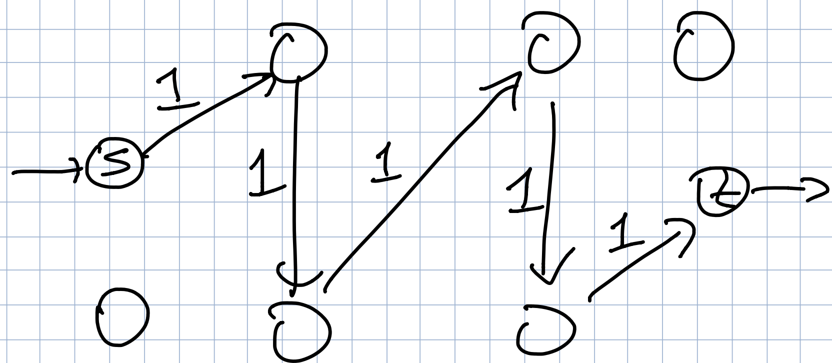


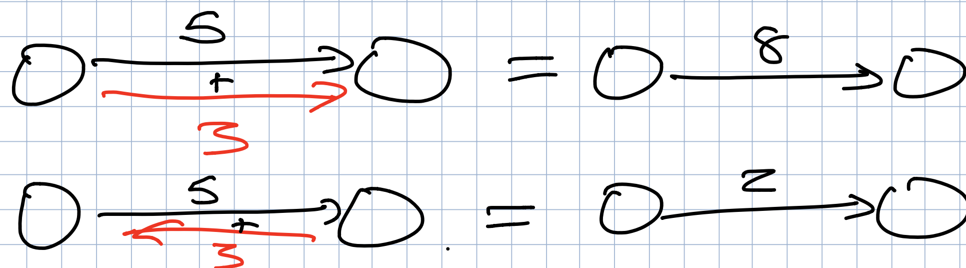
FF(G):

Find any flow F in G

$F' \leftarrow FF(G_F)$

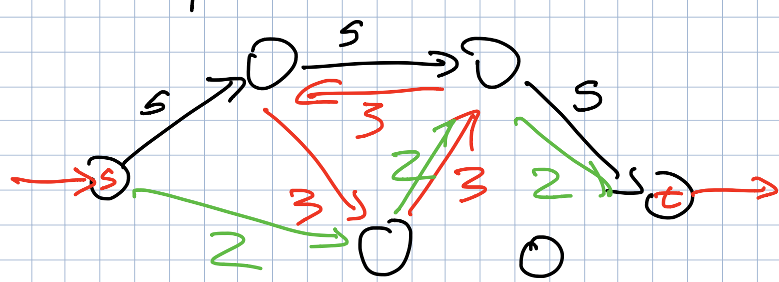
return $F + F'$





Flow decomposition:

Any flow F in any graph G can be written as a weighted sum of $s \rightsquigarrow t$ paths and cycles.

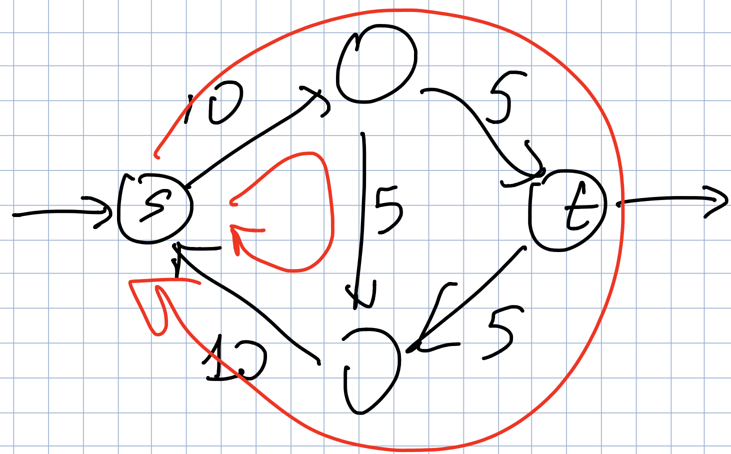


- If an edge $u \to v$ appears in any path or cycle then $F(u \to v) > 0$
- $\#\text{paths} + \#\text{cycles} \leq E - V + 1$
- $|F| = 0$ — circulation $= \sum$ cycles
- F is acyclic $= \sum$ paths from s to t

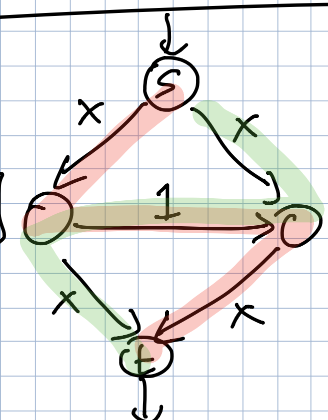
Cor: There is a max flow that is acyclic.

Cor: compute decomposition in $O(VE)$ time.

Any algo that incrementally computes max flows requires $\Omega(VE)$ time in worst case.



Smarter Ford Fulkerson



- "Fat pipe" [Edmonds Karp '72]
  push along path
  with max capacity
  Compute in $O(E \log E)$
- Assume int capacities $\text{maxflow}(G) = F$
- Let f^* be any max flow
  Decompose into $\leq E$ paths + cycles
  One of these paths has capacity $\geq \frac{F}{E}$

$\Rightarrow$ The fattest path has capacity $\geq \frac{F}{E}$

After we augment along that path

$$F \geq \text{maxflow}(G_f) + \frac{F}{E}$$

$$\boxed{\text{maxFlow}(G_f) \le \left(1 - \frac{1}{E}\right) \cdot \text{maxFlow}(G)}$$

$$\le e^{-1/E} \cdot \text{maxFlow}(G)$$

After k iterations:

$$\text{maxFlow}(G_f) \le e^{-k/E} \cdot F < 1$$

when $k = E \cdot \ln F + 1$

$$\boxed{\Rightarrow O(E^2 \log E \log F) \text{ time}}$$

"short pipes" – push flow on shortest paths

BFS

$O(E+V)$ time

$$\boxed{\text{Find shortest path from } s \text{ to } t \text{ in } G_f}$$

Intuition: over time nodes get further from s.

G_i = residual graph after i iterations

$\text{dist}_i(v)$ = dist from s to v in G_i

- $\text{dist}_i(v) \ge \text{dist}_{i-1}(v)$
- If ith iter saturates $u \to v$ then $\text{dist}_i(v) > \text{dist}_{i-1}(v)$

$\ge EV$ iterations $\Longrightarrow$ contradiction

Alg halts after at most EV iters

$\boxed{O(E^2V) \text{ time}}$