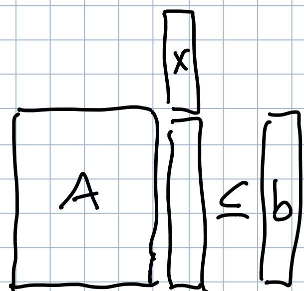
Simplex Algorithm [Dantzig '47]

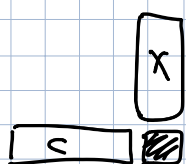
$$\boxed{\begin{array}{ll} \max & c\cdot x \\ \text{s.t} & Ax \le b \\ & x \ge 0 \end{array}}$$

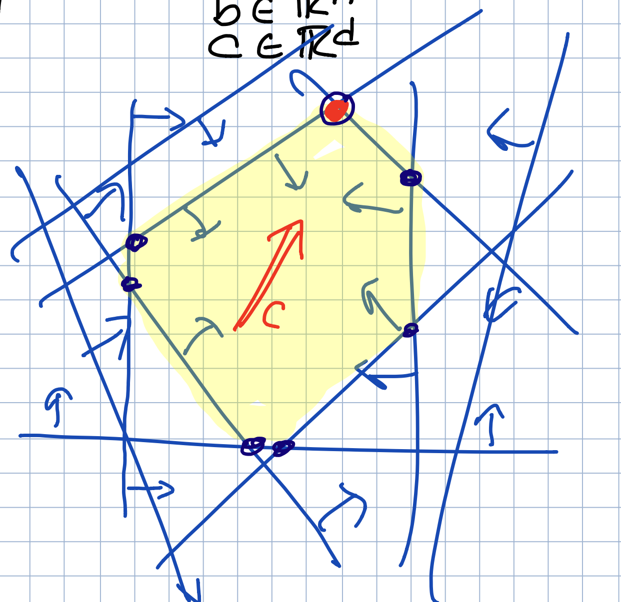
variables $x \in \mathbb{R}^d$

input $A \in \mathbb{R}^{n \times d}$
$b \in \mathbb{R}^n$
$c \in \mathbb{R}^d$

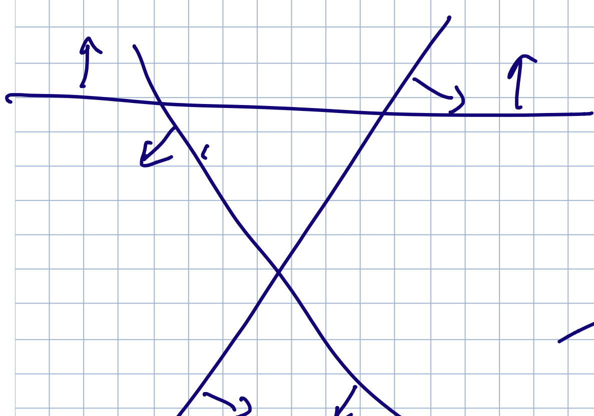


min



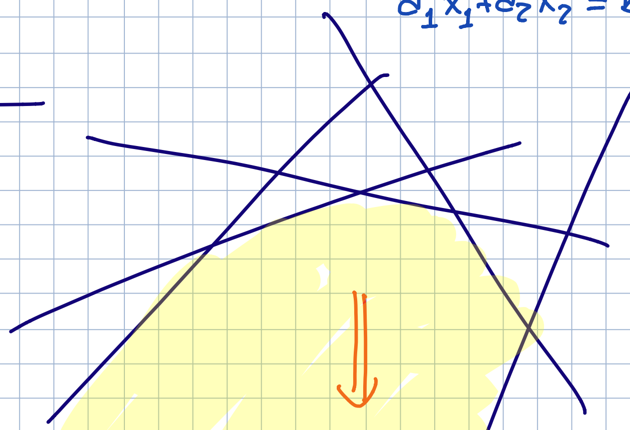


$a_1 x_1 + a_2 x_2 \le b$



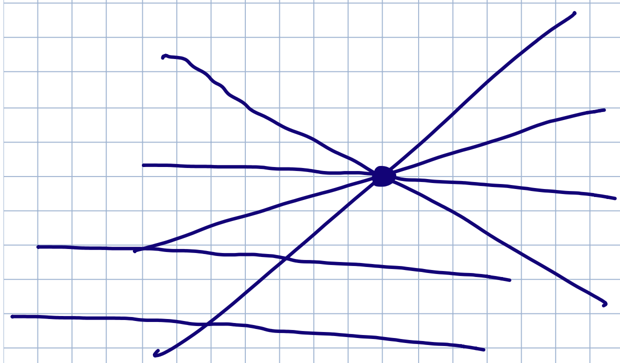
infeasible

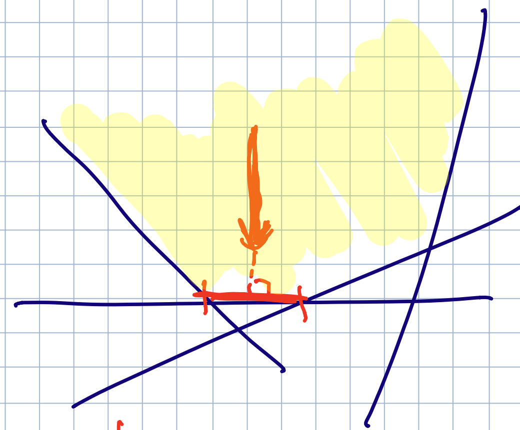
A b



unbounded

A c





nondegeneracy 1:

Constraints are linearly independent

A b

nondegeneracy 2:

Obj vector is lin. ind from constr.

A c

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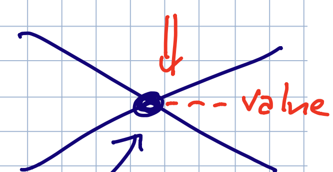
basis = subset of d constraints

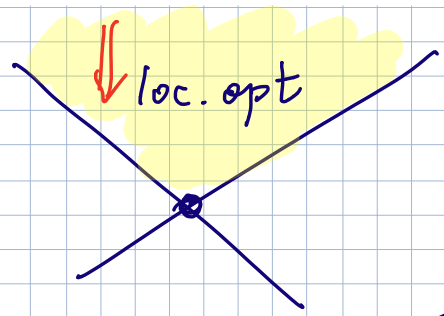


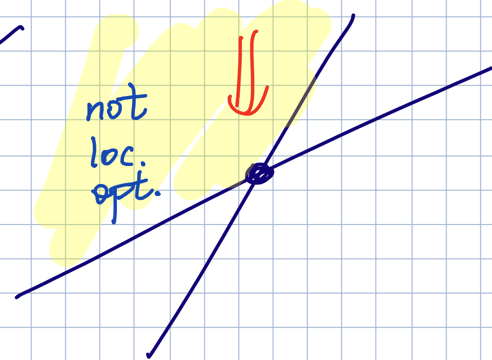
If LP is feasible and bounded then some basis is optimal

Try all $\binom{n+d}{d}$ bases $\longrightarrow$ algorithm!

A basis is feasible if loc satisfies all constraints

A basis is locally optimal if it is optimal for the smaller LP with only those constraints





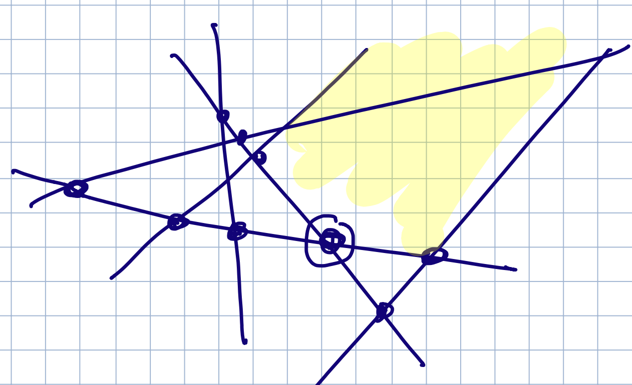
Duality Theorem for LP:

$$\boxed{\begin{array}{ll} \max & cx \\ \text{s.t} & Ax \le b \\ & x \geq 0 \end{array}} \qquad \boxed{\begin{array}{ll} \min & yb \\ \text{s.t} & yA \geq c \\ & y \geq 0 \end{array}}$$

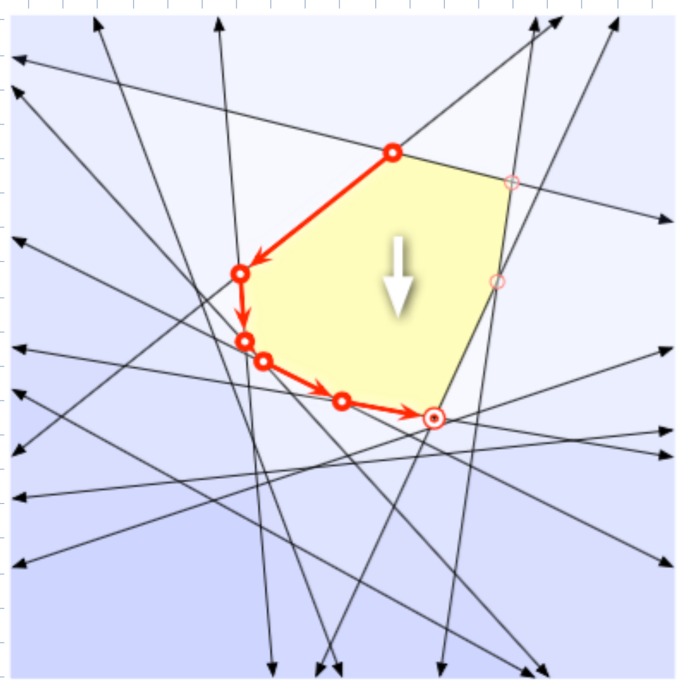
$$\binom{n+d}{d} = \binom{d+n}{n}$$

basis	$\rightleftarrows$	basis
feasible	$\rightleftarrows$	locally optimal
locally opt	$\rightleftarrows$	feasible
optimal	$\rightleftarrows$	optimal
neighbors	$\rightleftarrows$	neighbors

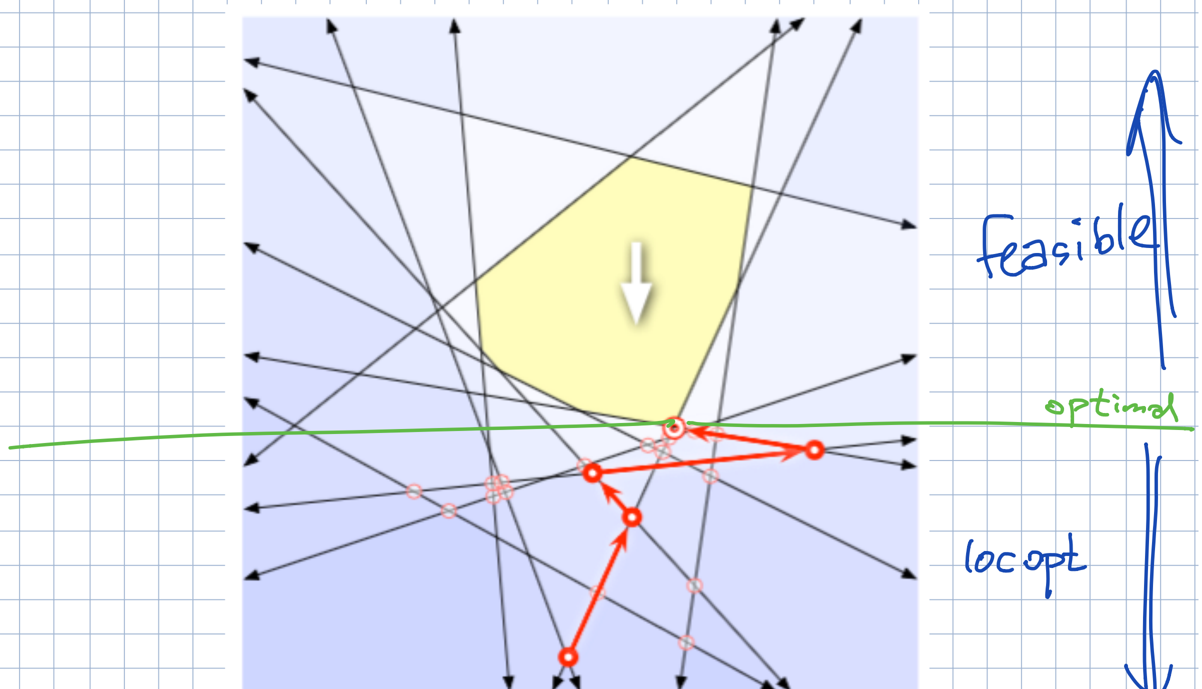
Basis B and basis B' are neighbors if $|B \cap B'| = d-1$



```
PRIMALSIMPLEX(H):
  if ∩H = ∅
      return INFEASIBLE
  x ← any feasible vertex
  while x is not locally optimal
      ⟨⟨pivot downward, maintaining feasibility⟩⟩
      if every feasible neighbor of x is higher than x
          return UNBOUNDED
      else
          x ← any feasible neighbor of x that is lower than x
  return x
```



```
DualSimplex(H):
  if there is no locally optimal vertex
      return Unbounded
  x ← any locally optimal vertex
  while x is not feasbile
      ⟨⟨pivot upward, maintaining local optimality⟩⟩
      if every locally optimal neighbor of x is lower than x
          return Infeasible
      else
          x ← any locally-optimal neighbor of x that is higher than x
  return x
```

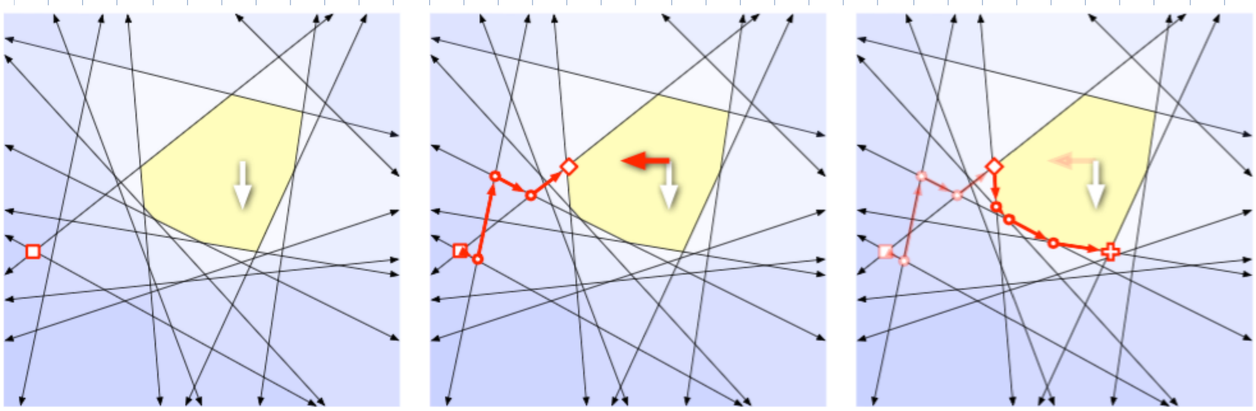


$$c \cdot x \leq y \cdot A \cdot x \leq y \cdot b$$

```
DUALPRIMALSIMPLEX(H):
  x ← any vertex
  H̃ ← any rotation of H that makes x locally optimal

  while x is not feasible
      if every locally optimal neighbor of x is lower (wrt H̃) than x
          return INFEASIBLE
      else
          x ← any locally optimal neighbor of x that is higher (wrt H̃) than x

  while x is not locally optimal
      if every feasible neighbor of x is higher than x
          return UNBOUNDED
      else
          x ← any feasible neighbor of x that is lower than x
  return x
```

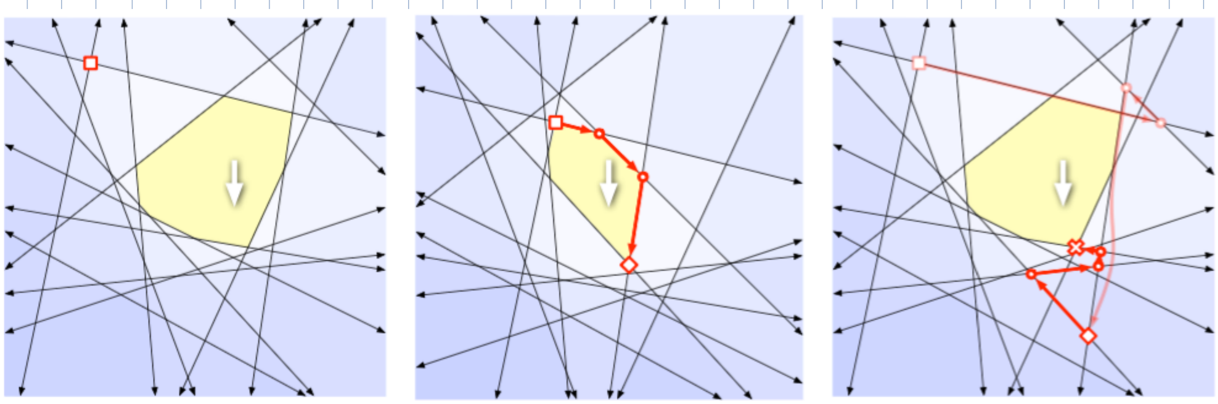


(a) (b) (c)

```
PrimalDualSimplex(H):
    x ← any vertex
    H̃ ← any translation of H that makes x feasible

    while x is not locally optimal
        if every feasible neighbor of x is higher (wrt H̃) than x
            return Unbounded
        else
            x ← any feasible neighbor of x that is lower (wrt H̃) than x

    while x is not feasible
        if every locally optimal neighbor of x is lower than x
            return Infeasible
        else
            x ← any locally-optimal neighbor of x that is higher than x
    return x
```



(a) (b) (c)

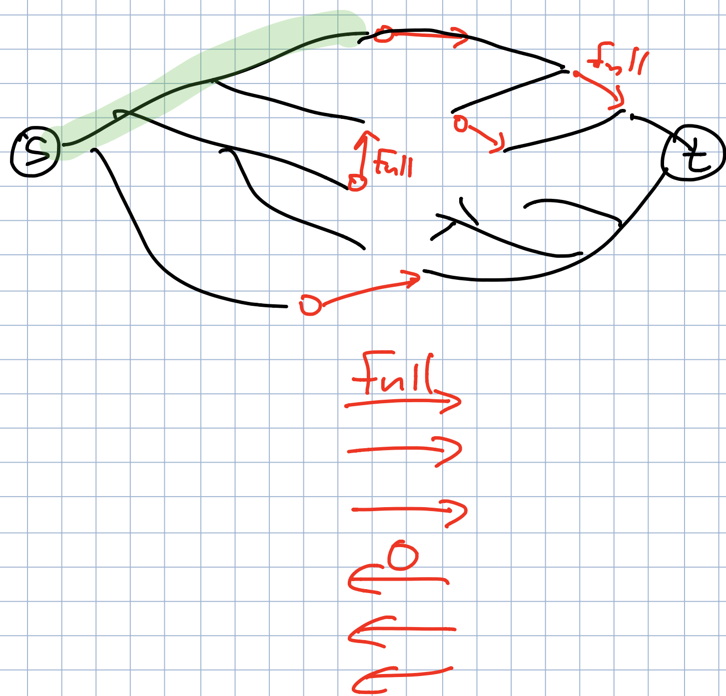
Running time: $O(n^{\lfloor d/2 \rfloor})$

OPEN Problem:

Can arbitrary LPs be solved in poly time?

Closed problem:

If all A, b, c are integers then LP can be solved in poly(#bits(A, b, c)) time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