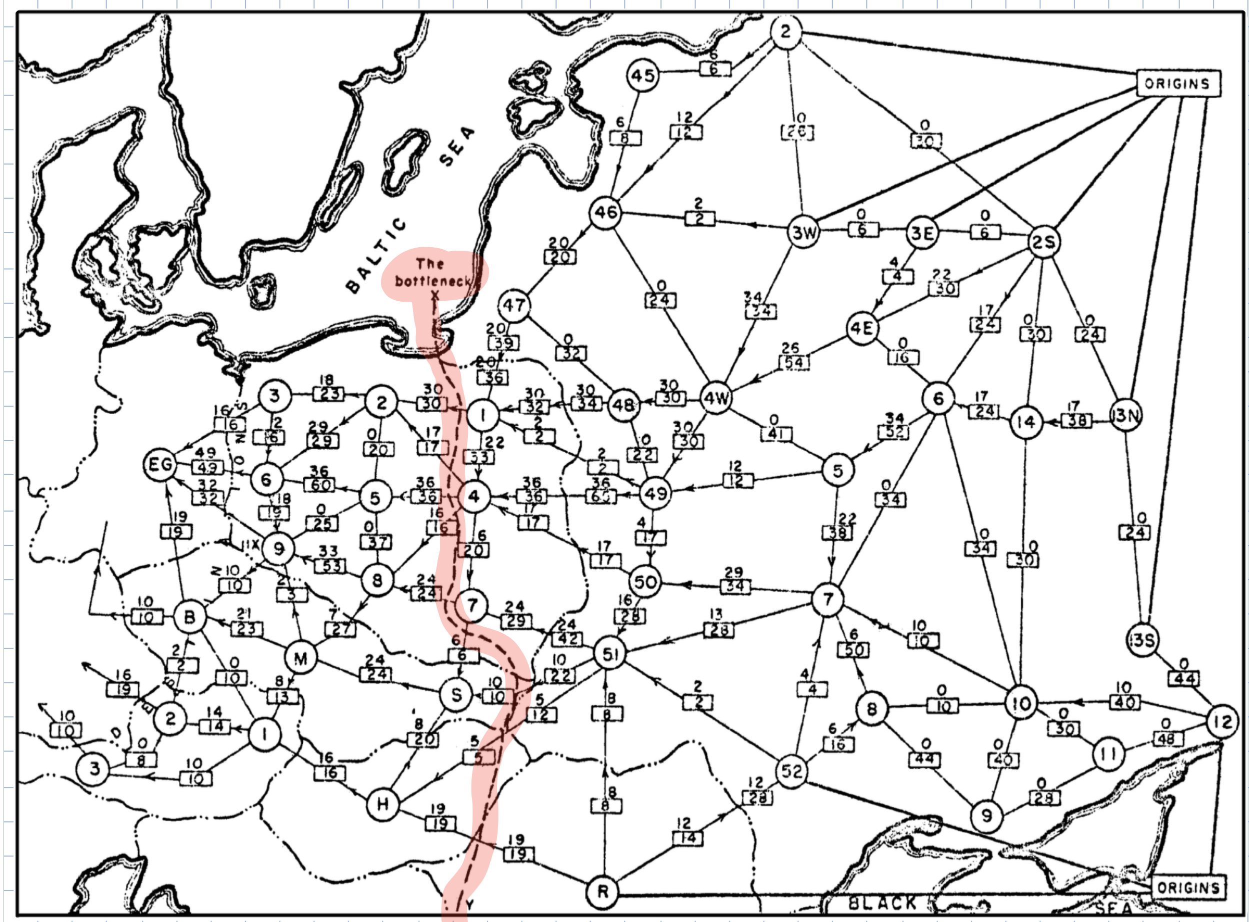
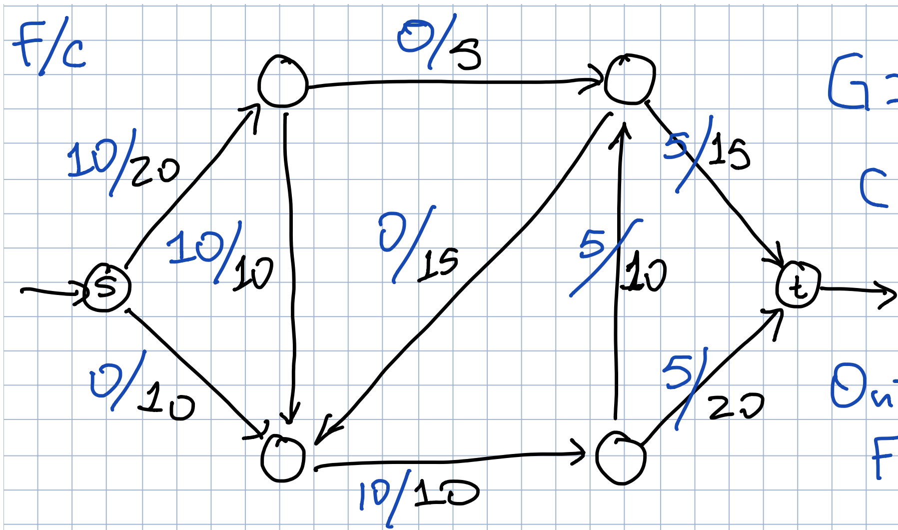


Harris and Ross 1950 (1999)

Maximum Flow   Minimum Cut

f/c



$G = (V, E)$

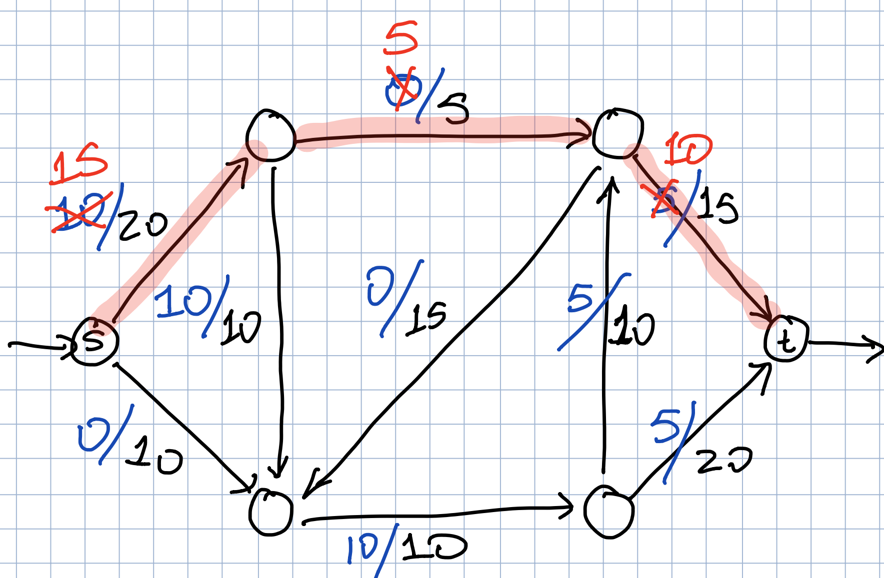
$c : E \to \mathbb{R}^+$

Output:
$f : E \to \mathbb{R}^{\geq 0}$

capacity: $0 \leq f(u \to v) \leq c(u \to v)$

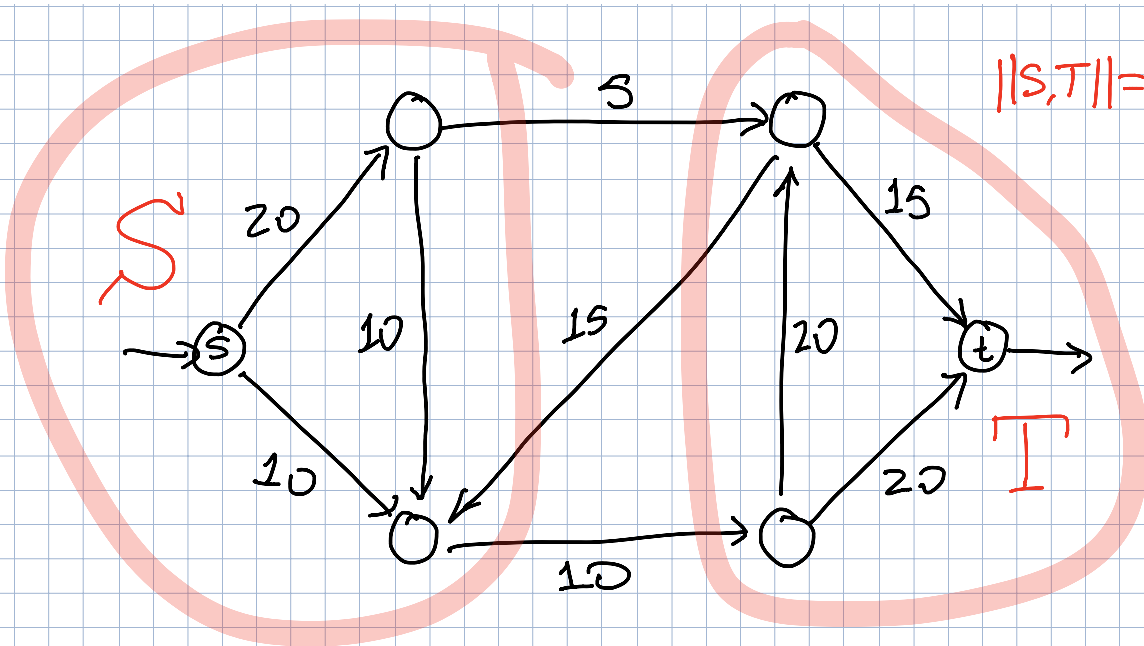
conservation balance $\sum_u f(u \to v) = \sum_w f(v \to w)$ for all $v \neq s, t$

maximize $|f| = \sum_w f(s \to w) - \sum_u f(u \to s)$



If $|f| = \|S, T\|$, then f is max flow
S, T is min cut

$\|S,T\| = 15$



(s,t)-cut = partition of vertices into $S \ni s$ and $T \ni t$

$$\text{capacity } \|S,T\| := \sum_{u \in S} \sum_{v \in T} c(u \to v)$$

Lemma: Let f be *any* flow
S,T be *any* cut
Then $|f| \le \|S,T\|$.

Proof:

$\|S,T\| \leftarrow |f|$

$$|f| = \sum_w f(s \to w) - \sum_u f(u \to s) \qquad \text{def } |f|$$

$$= \sum_{v \in S} \left(\sum_w f(v \to w) - \sum_u f(u \to v) \right) \qquad \text{balance}$$

$$= \sum_{v \in S} \left(\sum_{w \in T} f(v \to w) - \sum_{u \in T} f(u \to v) \right) \qquad \text{remove dupes } \pm$$

$$\le \sum_{v \in S} \sum_{w \in T} f(v \to w) \qquad f \ge 0$$

$$\le \sum_{v \in S} \sum_{w \in T} c(v \to w) \qquad f \le c$$

$$= \|S,T\| \qquad \text{def } \|S,T\|$$

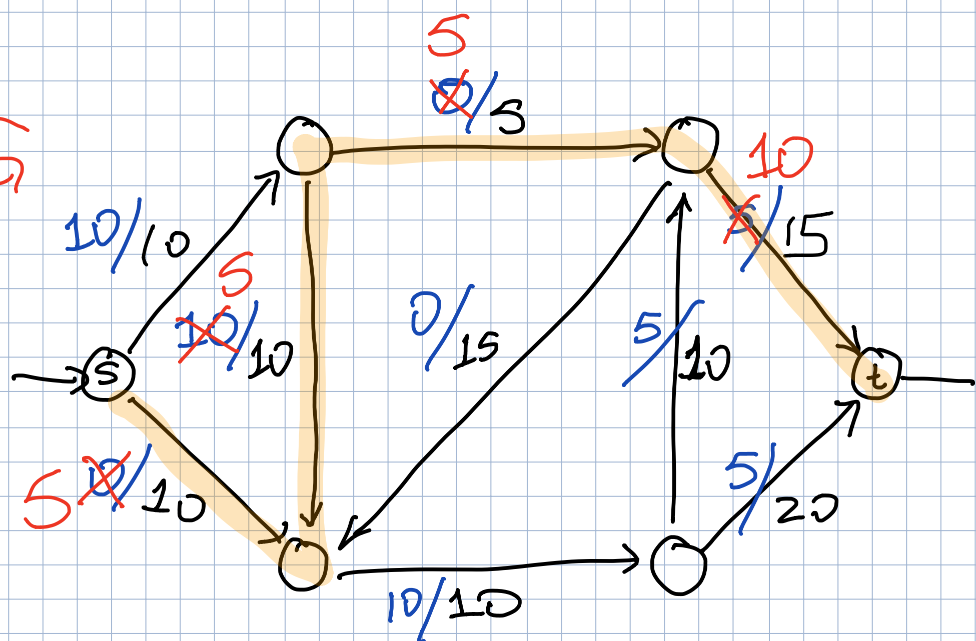
Corollary:

$$f(u \to v) = \begin{cases} 0 & \text{if } u \in T \ v \in S \\ c(u \to v) & \text{if } u \in S \ v \in T \end{cases}$$

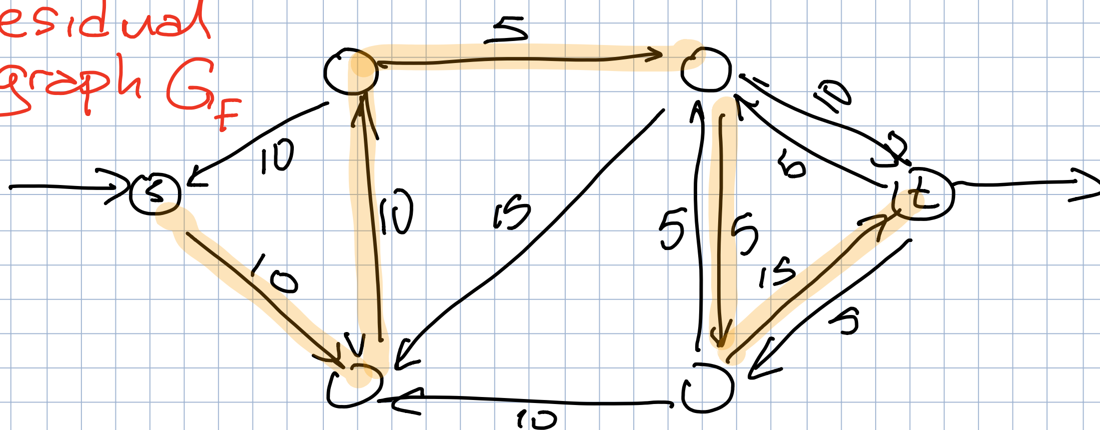
$\boxed{\text{IFF}}$ $|f| = \|S,T\|$

f is max S,T is min cut

G



Residual graph G_f



$$c_f(u\to v) = \begin{cases} c(u\to v) - f(u\to v) & \text{if } u\to v \in E \\ f(v\to u) & \text{if } v\to u \in E \end{cases}$$

If G_f has a path from s to t: $O(V+E)$ time

f is not max flow

Let $F = \min c_f(u\to v)$ along path

$f' \leftarrow f + F \cdot \text{path}$

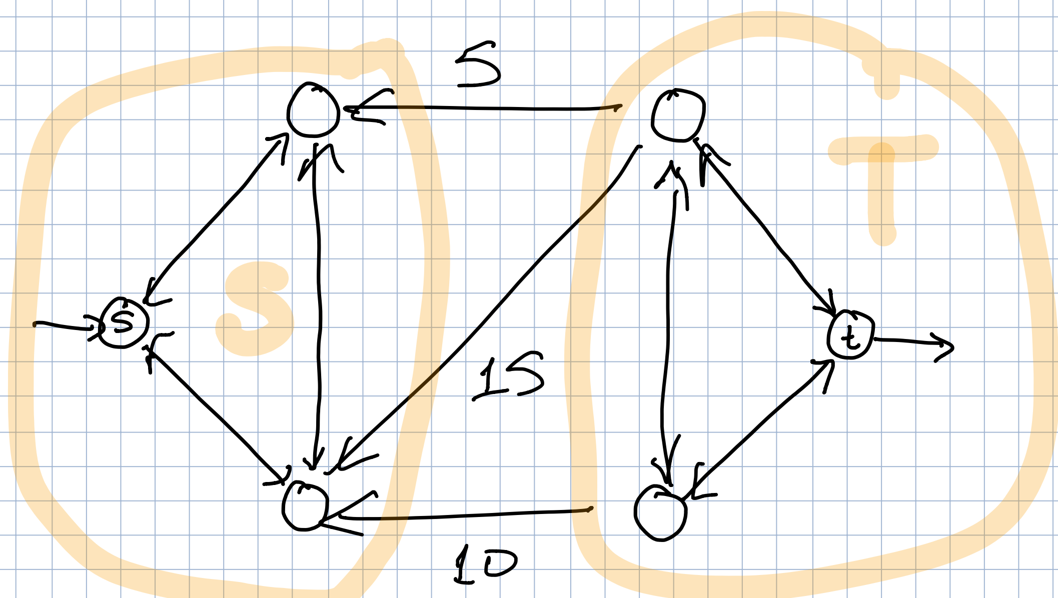
Ford Fulkerson

$f \leftarrow 0$

while G_f has path from s to t

augment f along that path

return f



if no $s \rightsquigarrow t$ path

$S = Reach(s) \quad T = V \setminus S$

All edges $S \to T$ full

$T \to S$ empty

$\Rightarrow$ f is maxflow and S,T are min cut

① If all capacities are ints

Every iter increases $|f|$ by ≥ 1

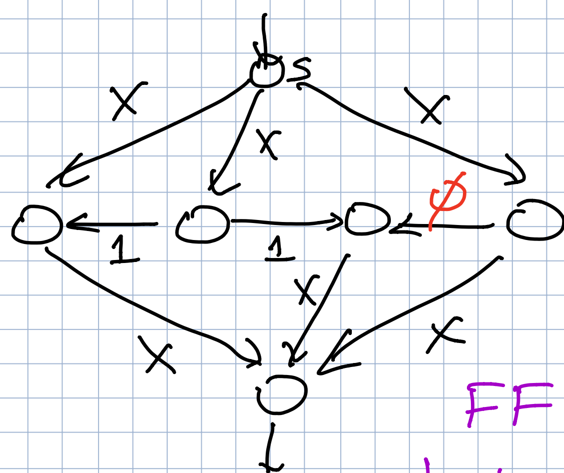
$\Rightarrow$ #iters $\leq |f^*|$

↑ output

in worst case,

exponential!

② If capacities are real #s



$2X+1$

FF ∞ loops

$|f| \longrightarrow 4+\sqrt{5}$

< 7