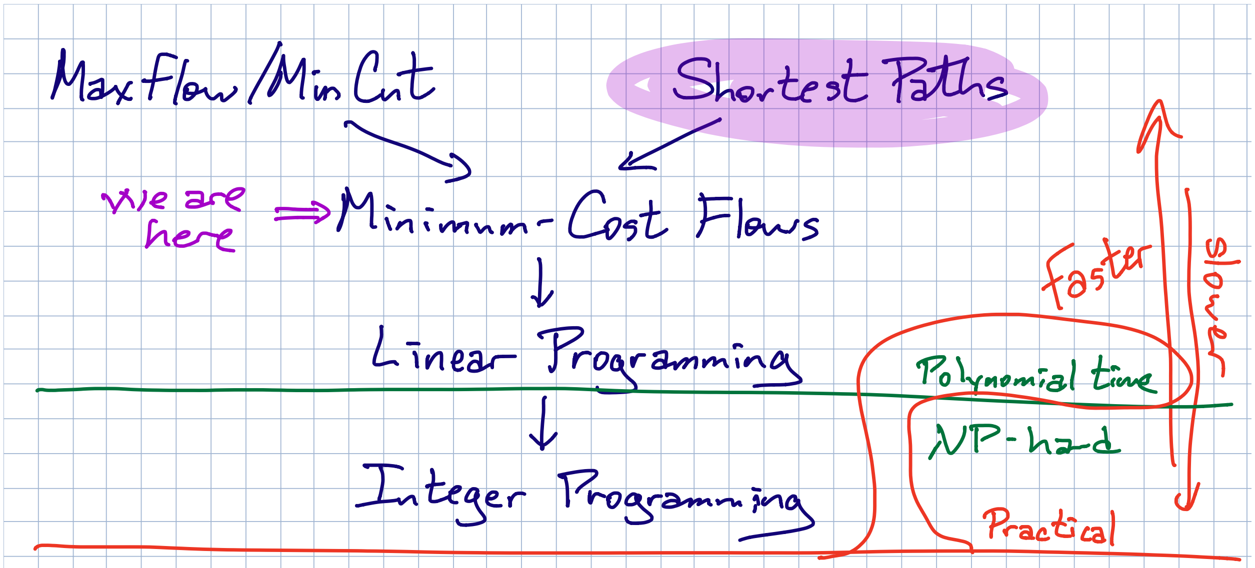
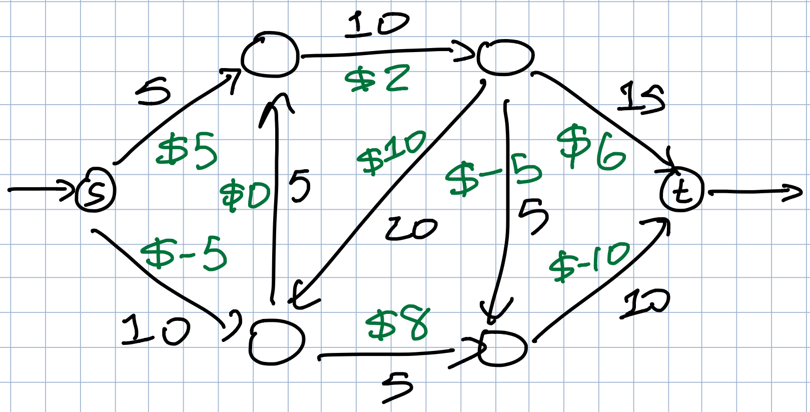




Max-Flow: Edges have capacities c(u→v)

maximize value

Min-cost Flow: Edges also have costs $(u→v)

minimize total cost:

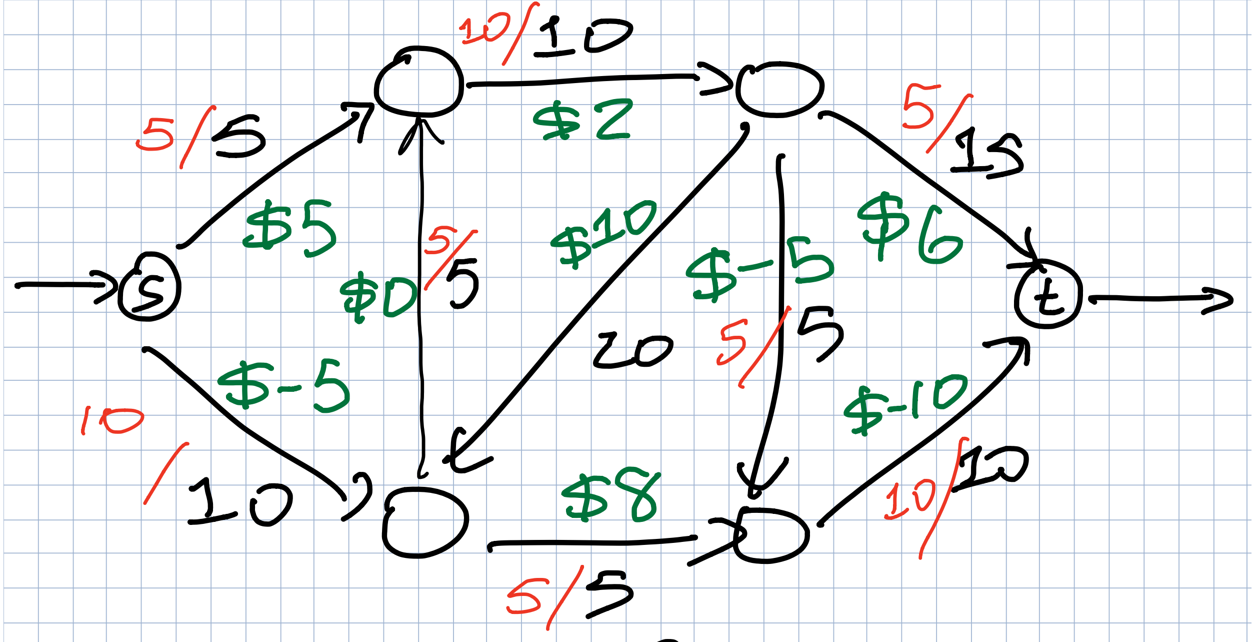
$$\boxed{\sum_{u\to v} f(u\to v)\cdot \$(u\to v)}$$

and max value.

General strategy

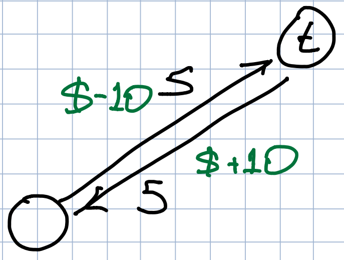
① Find <u>any</u> max flow

② Incrementally improve it



Residual cap $c_f(u\to v) = \begin{cases} c(u\to v) - f(u\to v) & \text{if } u\to v \in E \\ f(v\to u) & \text{if } v\to u \in E \end{cases}$

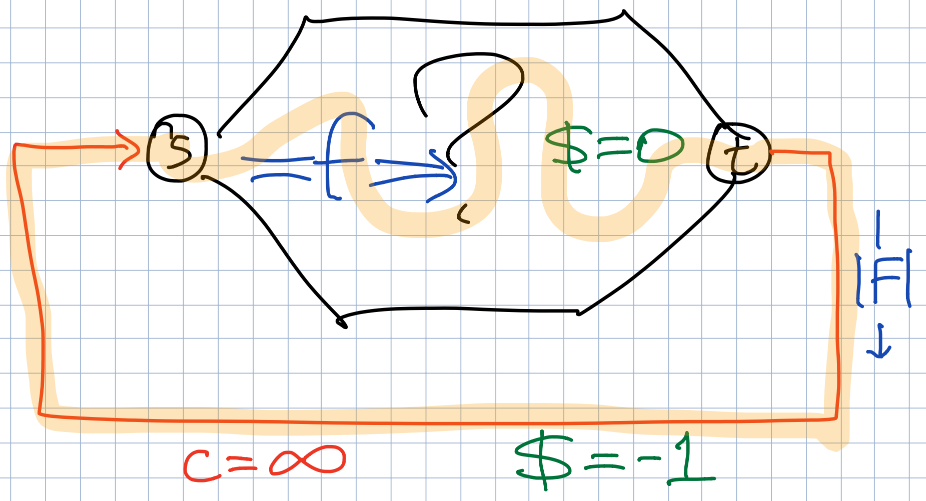
Residual cost $\$_f(u\to v) = \begin{cases} \$(u\to v) & \text{if } u\to v \in E \\ -\$(v\to u) & \text{if } v\to u \in E \end{cases}$



F is a min-cost flow $\iff$ $G_f$ has no directed cycles with negative total $\$_f$

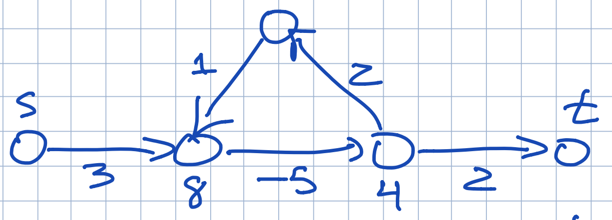
① Find any max flow

② While $G_f$ has a neg cost cycle
push flow around neg cycle



$\$ = -|f|$

Ford Fulkerson ⊆ cycle canceling

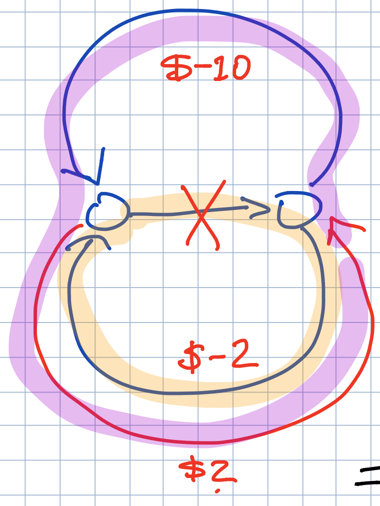


Neg cycles mean no shortest paths

Find Neg cycles using shortest path algo

Modification of Bellman Ford

Finds neg cycle in O(VE) time



How many iterations?

integer caps

integer costs

each iter reduces $(F)

by $\geq 1$

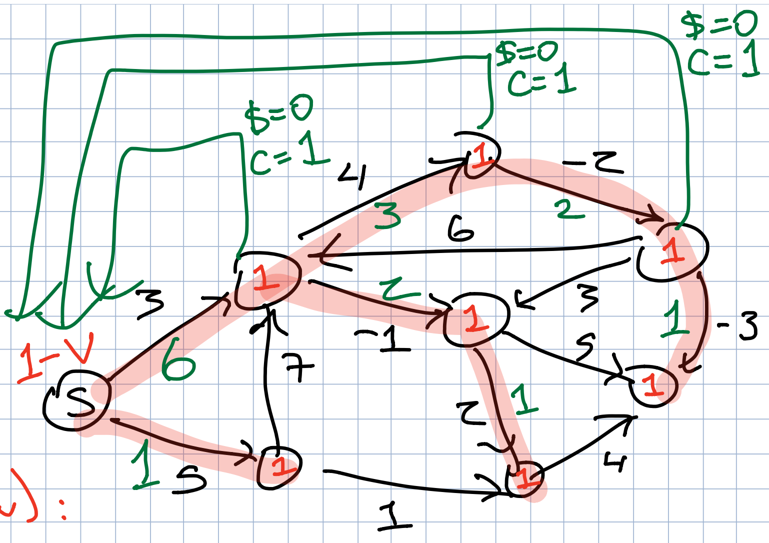
$\Rightarrow$ #iters $\leq C \cdot D$

$C = \sum_e c(e) \quad D = \sum_e |\$(e)|$

min mean cycle

minimize $\frac{\text{total cost of cycle}}{\text{\#edges in cycle}}$

$\Rightarrow O(VE^2 \log^2 V)$ time overall



balance b(v):

$$\sum_u F(u\to v) - \sum_w F(v\to w) = b(v)$$

$$\$(e) = w(e)$$

$$c(e) = \infty$$

Consider a min-cost circulation problem

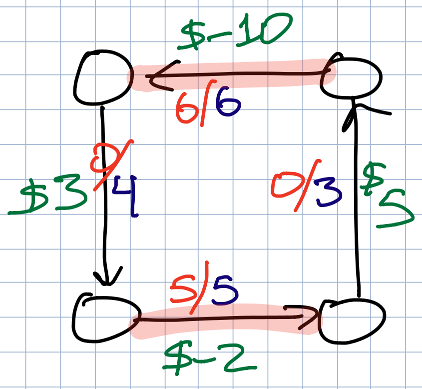
G graph cap c cost $

Past: Find any circ —— remove neg cycles

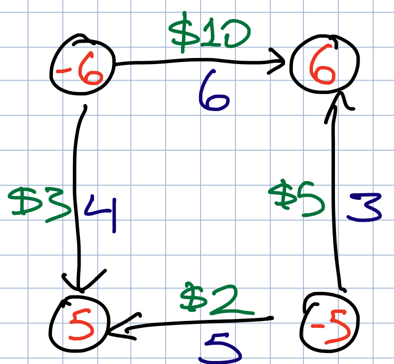
balance

Now: remove all neg cycles —— restore balance

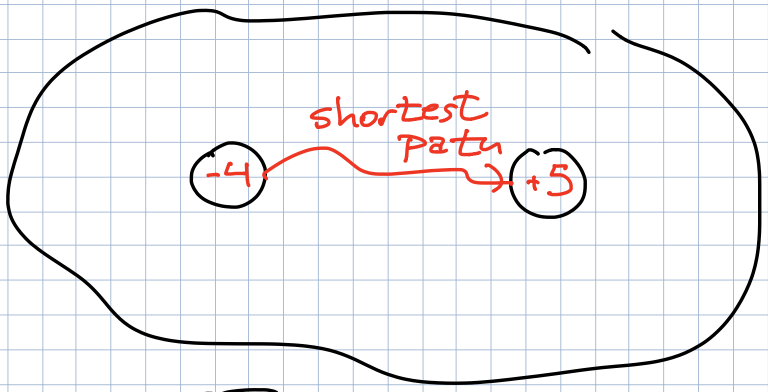
??



saturate all neg edges



residual graph with res. balances



Successive SP:

Find any node u s.t $b_f(u) < 0$

v s.t $b_f(v) > 0$

push $\underbrace{\text{flow}}_{c_f}$ along $\underbrace{\text{shortest}}_{\$_f}$ path $u \to v$

$\Uparrow$ O(VE) BF

$O(E^2 \log^2 V)$ time [Orlin]