

$T_A(x)$ - time for A
on input X

$T_A(n) = \max_{|X|=n} T_A(x)$ - worst-case
time

$E_{|X|=n} [T_A(x)] =$ "average"
time

A is randomized

$\max_{|X|=n} E[T_A(x)] =$ worst case
expected
time

RANDOM(k)

returns element of
 $\{1, 2, \dots, k\}$ uniformly and
independently at random

Find one matching pair

worst case: $\leq n-1$ tests

randomized:

Pick a random bolt

For $i \leftarrow 1$ to $n-1$

try nut i

if match return i

return n

$$\text{Exp \# tests} = \frac{n+1}{2} - \frac{1}{n} \quad ?$$

$$E[T(n)] = \sum_{t=1}^{n-1} t \cdot \Pr[T(n)=t]$$

$$\Pr[T(n)=t] = \begin{cases} 1/n & \text{if } t < n-1 \\ 2/n & \text{if } t = n-1 \end{cases}$$

$$\Pr[T(n)=3] = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2}$$

Match(n):

pick a random bolt B
 $2n-1$ { split nuts into
 $< B$ $N=B$ $> B$
 split bolts into
 $< N$ $> N$

Match($< B$, $< N$)

Match($> B$, $> N$)

$$\bar{T}(n) = 2n - 1 + \mathbb{E}_k \left[\bar{T}(k-1) + \bar{T}(n-k) \right]$$

Linearity of expectation

$$E[X+Y] = E[X] + E[Y]$$

good pivot: $\frac{n}{4} \leq k \leq \frac{3n}{4}$

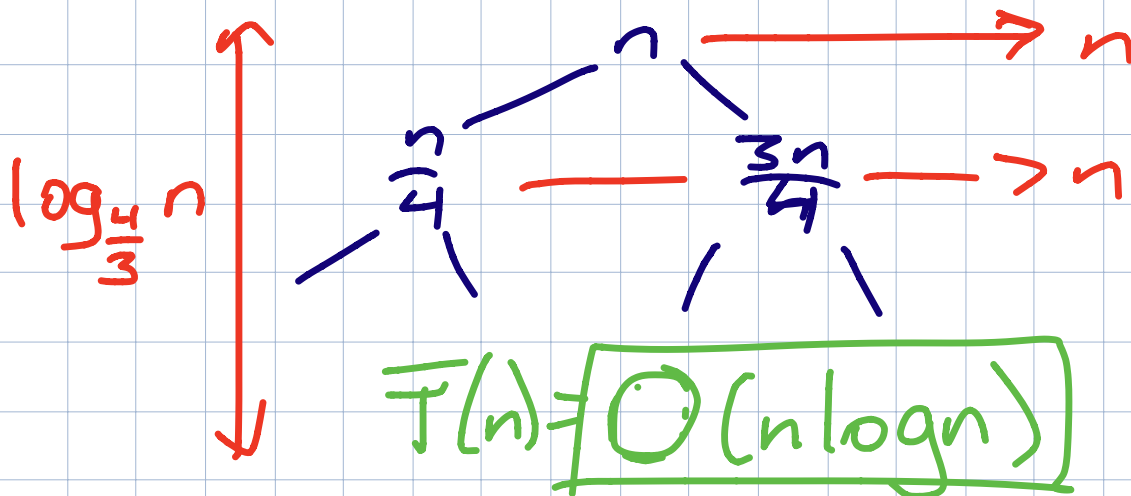
bad pivot: otherwise

$$\Pr[\text{good}] = \frac{1}{2}$$

$$\bar{T}(n) = 2n - 1 + \mathbb{E}_k [\bar{T}(k-1) + \bar{T}(n-k)]$$

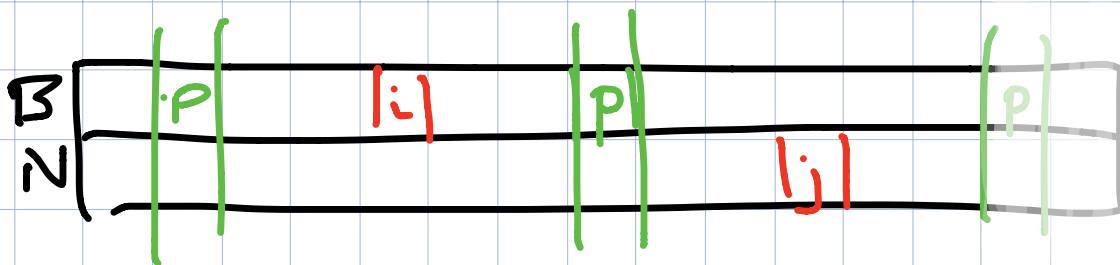
$$\frac{1}{2} \bar{T}(n) \leq 2n - 1 + \cancel{\frac{1}{2} \bar{T}(n)} + \frac{1}{2} [\bar{T}(\frac{n}{4}) + \bar{T}(\frac{3n}{4})]$$

$$\bar{T}(n) \leq \underbrace{(4n - 2)}_{O(n)} + \bar{T}(\frac{n}{4}) + \bar{T}(\frac{3n}{4})$$



$$E[T(n)] = \sum_{i=1}^n \sum_{j=1}^n P_r [\text{test bolt } i \text{ against nut } j]$$

$$E[X] = P_r[X=1]$$



$$P_r[\text{Test } i:i] = 1$$

$$P_r[\text{Test } 1:n] = 2/n$$

Test $i:j \iff$ First pivot in $\{i, i+1, \dots, j\}$

$$P_r[\text{test } i:j] = \frac{2}{|j-i+1|}$$

is i or j

$$\begin{array}{cccccccc}
 1 & 1 & 1 & 1 & \dots & - & - & \dots & 1 & 1 \\
 1 & 1 & 1 & 1 & \dots & \dots & \dots & \dots & 1 & \\
 \frac{1}{2} & \frac{1}{2} & - & - & - & - & - & - & \frac{1}{2} & \\
 \frac{1}{3} & - & - & \dots & \dots & \dots & \dots & \dots & \frac{1}{3} & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 \frac{1}{n} & & & & & & & & &
 \end{array}$$

$$4nH_n - 7n + 4H_n$$

$$\uparrow H_n = \sum_{i=1}^n \frac{1}{i} \approx \ln(n+1)$$