How to tell if G has unique max flow?

1. Max Flow f^*
2. Construct G_{f^*}

Claim: f^* is unique $\iff$ G_{f^*} is a dag

If G_{f^*} has a cycle $\Rightarrow$ push flow around it

Another max flow f' $\Rightarrow$ $f' - f^*$ is a feasible circulation in G_{f^*}

flow decomposition theorem

$\Rightarrow f' - f^* = \sum$ cycles in G_{f^*}

$\Rightarrow G_{f^*}$ not a dag

Flow Decomposition Theorem

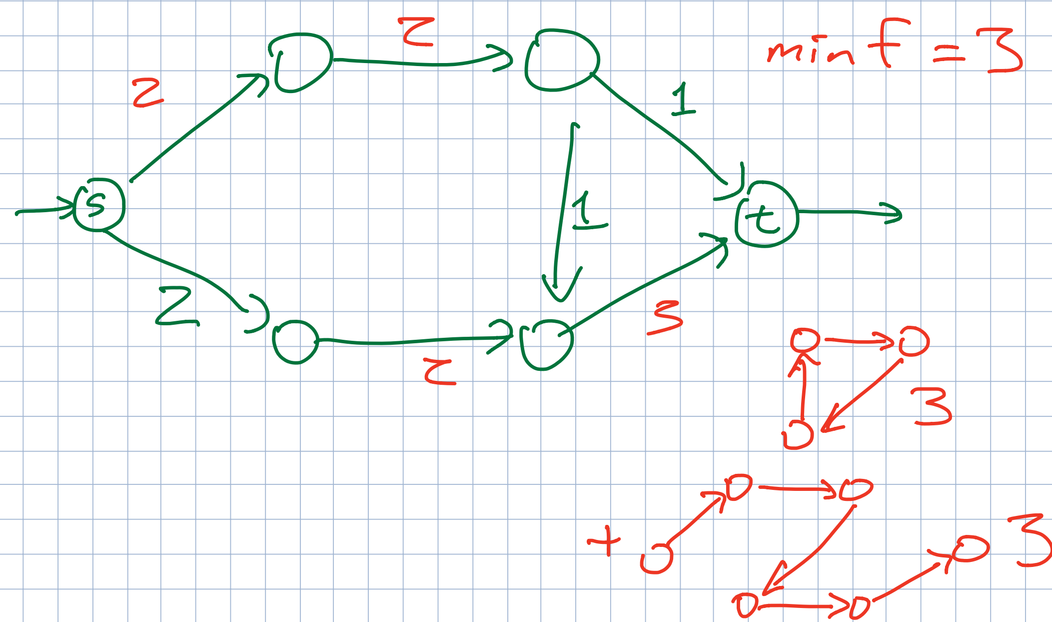
Any (s,t)-flow f in G

$$\underline{f} = \sum \underset{\text{flows}}{\overset{s \to t}{\text{path}}} + \sum \underset{\text{flows}}{\text{cycle}}$$

Every edge in every path + cycle carries pos. flow $\underline{f}(u \to v) > 0$

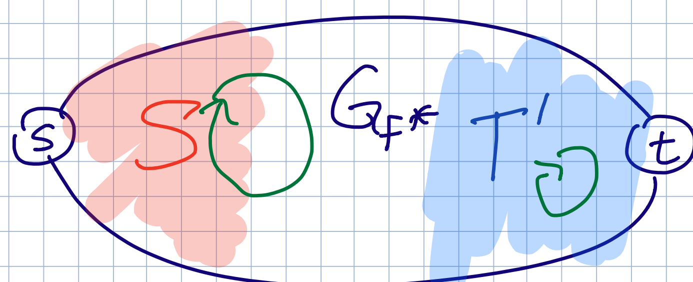
$\#\text{paths} + \#\text{cycles} \le E$

Algo: compute decomp in $O(VE)$ time



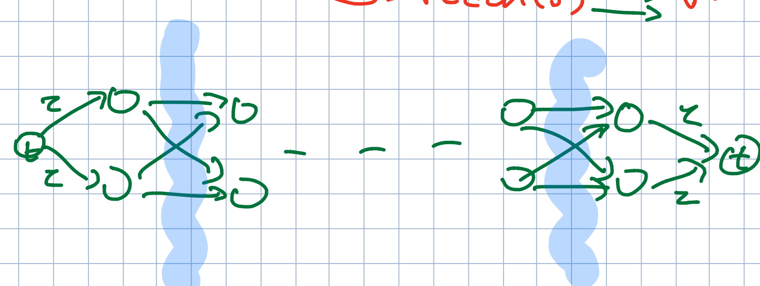
Unique min cut?

(1) Max Flow F^* (2) Compute G_{f^*}



$S' = V \setminus T'$ $T' = \text{reach}^{-1}(t)$

$S = \text{reach}(s) \Rrightarrow T = V \setminus S$



Universal Hashing $\Pr_h[h(x)=h(y)] \le 1/m$ for all $x \ne y$

n items $\longrightarrow$ size $m = 2n$ assume $\sqrt{n}$ int

(a) Prove $E[\#\text{collisions}] \le \frac{n}{4}$

$$E[\#\text{collisions}] = \sum_{\{x,y\}} \Pr[h(x)=h(y)]$$

$$\le \sum_{\{x,y\}} 1/m = \binom{n}{2}/m = \frac{n(n-1)}{2\cdot 2n}$$

$$= \frac{n-1}{4} \le \frac{n}{4} \quad \square$$

(b) $\Pr[\#\text{coll} \ge \frac{n}{2}] \le 1/2$

$$\hookrightarrow \le \frac{E[\#\text{coll}]}{n/2} \quad [\text{Markov}]$$

$$\le \frac{n/4}{n/2} = \frac{1}{2} \quad \square$$

(c) $\Pr\left[\begin{array}{l}\text{some subset of} > \sqrt{n} \text{ items} \\ \text{all hash to same value}\end{array}\right] \le 1/2.$

$$\hookrightarrow \le \Pr\left[\#\text{coll} \ge \binom{\sqrt{n}+1}{2}\right] \le \Pr\left[\#\text{coll} \ge \frac{n}{2}\right] \le 1/2 \quad \square$$

$$\binom{\sqrt{n}+1}{2} = \frac{(\sqrt{n}+1)\sqrt{n}}{2} \ge \frac{n}{2}$$

(d) Assume h is 4-uniform

Prove $\Pr[>\sqrt{n} \to \text{one hash value}] = O\left(\frac{1}{n}\right)$

$\hookrightarrow$ events $[h(x)=h(y)]$ are pairwise indep

$\Rightarrow$ Chebyshev's $\ne$

$$\Pr\left[\#\text{coll} \geq \tfrac{n}{2}\right] \leq \frac{1}{E[X]} = \frac{4}{n-1} = O\left(\frac{1}{n}\right)$$

$$\Pr\left[X \geq (1+\delta)E[X]\right] \leq \frac{1}{\delta^2 E[X]}$$

Mult hashing

pick a, b at random $a, b \in [p]$ p is prime

$$h_{a,b}(x) = (ax + b \bmod p) \bmod m$$

universal $\Pr[h(x) = h(y)] \leq 1/m$ for all $x \neq y$

$\Uparrow$

2-uniform $\Pr[h(x) = i \text{ and } h(y) = j] = 1/m^2$

for all $x \neq y$ and all i, j

$\Uparrow$

4-uniform

$$E[\#\text{coll}] = \sum_{x \neq y} \Pr[h(x) = h(y)]$$

#full

$n \le m/2$ ideal random h open addressing

probe a random permutation of indices until we find empty

$X_i = \#$ probes for i^{th} insertion

$X = \max X_i$

(a) $\Pr[X_i > k] \le 1/2^k$ for all i and k

$\Pr[k \text{ random addresses all full}] = \left(\frac{n}{m}\right)\cdot\left(\frac{n-1}{m-1}\right)\cdots$

$= \left(\Pr[\text{address is full}]\right)^k$

$\le \left(\frac{1}{2}\right)^k$ $\square$

(b) $\Pr[X_i > 2\lg n] \le 1/n^2$ for all i.

$k = 2\lg n$ $\square$

(c) $\Pr[X > 2\lg n] \le 1/n$

Union bound. $\square$

$\Pr[X > 2\lg n] = \Pr[X_1 > 2\lg n \vee X_2 > 2\lg n \vee \cdots \vee X_n > 2\lg n]$

$\le \Pr[X_1 > 2\lg n] + \Pr[X_2 > 2\lg n] + \cdots$

$\le 1/n.$

(d) $E[X] = O(\log n)$

$E[X] = \sum_x x \cdot \Pr[X = x] = \sum_x \Pr[X \ge x]$

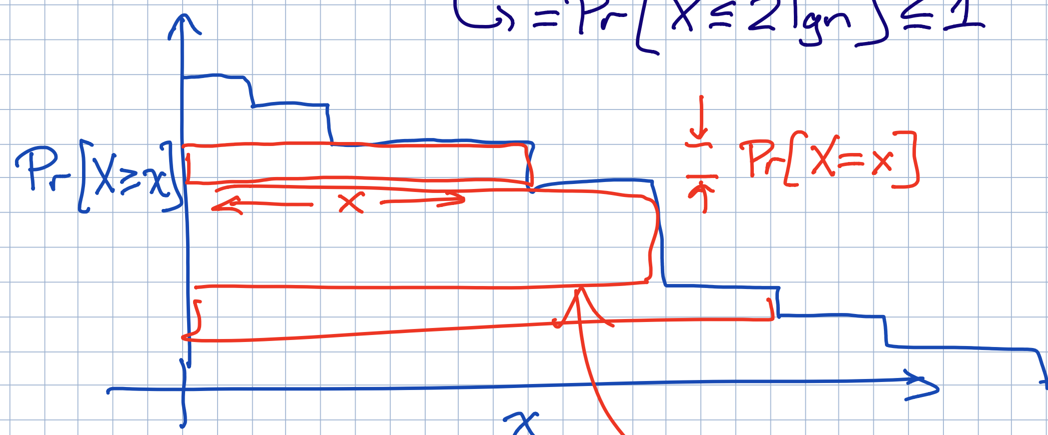
$\sum_{x=1}^{2\lg n} x \cdot \Pr[X = x] + \sum_{x=2\lg n+1}^{n} x \cdot \Pr[X = x]$

$$\leq 2\lg n \left(\sum_{x=1}^{2\lg n} \cdot \Pr[X=x] \right) + n \cdot \left(\sum_{x=2\lg n+1} \Pr[X=x] \right)$$

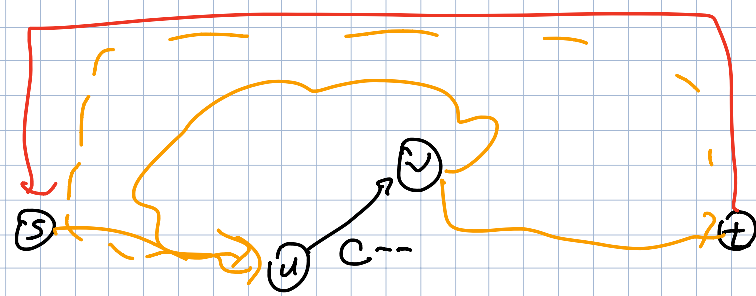
$$= \Pr[X \leq 2\lg n] \leq 1$$

$$\leq 2\lg n \cdot 1 + n \cdot \frac{1}{n} = 2\lg n + 1$$

$$= O(\log n)$$



$$\sum_x \Pr[X \geq x] = \text{Area} = E[X]$$



If $u \to v$ is in every min cut

Decreasing $c(u \to v)$ decreases $|f^*|$

Else

It doesn't

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value of ANY flow $\leq$ cap of ANY cut

value of MAX flow $=$ cap of MIN cut