Approximation alg: Given X

compute $A(X)$

s.t. $\frac{A(X)}{OPT(X)} \leq \alpha(X)$

$\ldots 2, \log|X|, \sqrt{n} \ldots$

Integer Linear Programming

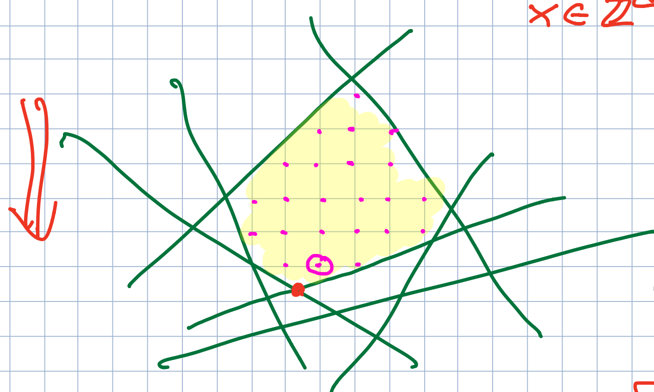
$\max \ c \cdot x$

s.t. $Ax \leq b$

$x \geq 0$

$x \in \mathbb{Z}^d$

sometimes redundant!



NP-hard!

Proof: Reduction from 3SAT

$(a \vee b \vee c) \wedge (\bar{a} \vee c \vee \bar{d}) \wedge \cdots$

$(a+b+c) \cdot (\bar{a}+c+\bar{d}) \cdots$

$$\boxed{\begin{array}{c} a+b+c \geq 1 \\ -a+c-d \geq -1 \\ \vdots \\ 0 \leq a, b, c \ldots \leq 1 \end{array}}$$

$a, b, c \ldots \in \mathbb{Z}$

Φ satisfiable

$\Updownarrow$

IP feasible.

$\square$

Min Vertex Cover

Input: undir $G=(V,E)$

Output: min # vertices in C

s.t. $\forall uv \in E$ either $u \in C$ or $v \in C$

Theorem: ILP is NP-hard

Proof: Reduction from Min VC

variables $\Longleftrightarrow$ vertices in G

$x_v = 1 \qquad v \in C$

$$\min \sum_v x_v \cdot w_v$$

$$x_u + x_v \geq 1 \quad \text{for all } uv \in E$$

$$\cancel{x_v \in \{0,1\} \quad \text{for all } v \in V}$$

$$0 \leq x_v \leq 1 \quad \text{for all } v \in V$$

OPT = ~~size~~ weight of min VC

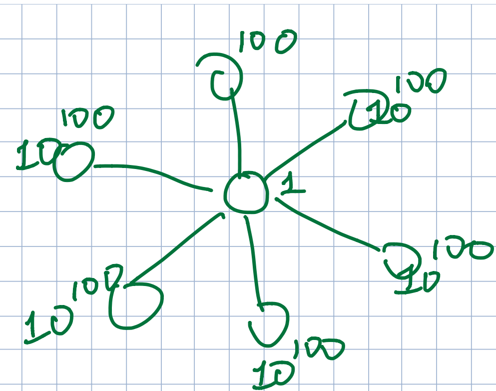
$\widetilde{OPT}$ = solution to LP relaxation
opt. fractional solution.

$$\boxed{OPT \geq \widetilde{OPT}}$$

$$x'_v = \begin{cases} 1 & \text{if } x^*_v \geq 1/2 \\ 0 & \text{if } x^*_v < 1/2 \end{cases}$$

Claim: x' describes a VC C'

Claim: $|C'| \leq 2 \cdot \widetilde{OPT} \leq 2 \cdot OPT$ $\square$

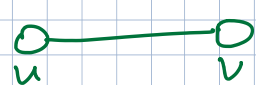


$$\boxed{\begin{array}{ll} \min \sum_v x_v \cdot w_v & \\ x_u + x_v \geq 1 & \text{for all } uv \in E \\ \cancel{x_v \in \{0,1\}} & \cancel{\text{for all } v \in V} \end{array}}$$

$$0 \leq x_v \leq 1 \quad \text{for all } v \in V$$

$$x'_v = \begin{cases} 1 & \text{with probability } x^*_v \\ 0 & \text{otherwise} \end{cases} \Bigg\} \underline{\text{indep}}$$

$$E\left[\sum_v x'_v \cdot w_v\right] = \sum_v E[x'_v] \cdot w_v = \sum_v x^*_v \cdot w_v = \widetilde{OPT}$$



$$\Pr[uv \text{ is covered}]$$
$$= 1 - \Pr[x'_u = 0] \cdot \Pr[x'_v = 0]$$
$$= 1 - (1 - x^*_u)(1 - x^*_v)$$
$$\geq 3/4$$

Run this experiment N times

N subsets $x'_1 \; x'_2 \; \cdots \; x'_N$ $\quad C = \bigcup_{i=1}^{N} x'_i$

$x'_{v,1} \quad x'_{v,2} \cdots x'_{v,N}$

$$E[w(C)] \le \sum_{i=1}^{N} E[w(x'_i)] = N \cdot \widetilde{OPT} \le N \cdot OPT$$

$$\Pr[uv \text{ is } \underline{\underline{\text{not}}} \text{ covered by } C] \le \left(\frac{1}{4}\right)^N \le \left(\frac{1}{4}\right)^{4 \lg n}$$

$$N = 4 \lg n \qquad\qquad = \frac{1}{n^8}$$

$$\Pr[C \text{ is } \underline{\text{not}} \text{ a vertex cover}] \le E/n^8 \le \frac{1}{n^6}$$

~~With high prob.~~ we get a vertex cover whose expected ~~size~~ weight is $O(\log n) \cdot OPT$. $\cancel{+ \frac{n^2}{n^6}}$