OLD CS 473: Fundamental Algorithms, Spring 2015

Discussion 15

May 7, 2015

15.1. Very very independent.

The following is a version of the independent Set Problem. You are given a graph $G = (V, E)$ and an integer k. For this problem, we will call a set $I \subset V$ strongly independent if, for any two nodes $v, u \in I$, the edge (v, a) does not belong to E, and there is also no path of two edges from u to v, that is, there is no node w such that both $(u, w) \in E$ and $(w, v) \in E$. The Strongly independent Set Problem is to decide whether G has a strongly independent set of size at least k.

Prove that the Strongly independent Set Problem is NP-Complete.

15.2. Recurrences.

Solve the following recurrences:

(A) $T(n) = T(\lfloor n/15 \rfloor) + T(\lfloor n/10 \rfloor) + 2T(\lfloor n/6 \rfloor) + n$
(B) $T(n) = T(n-1) + 2n - 1$, where $T(0) = 0$
(C) $T(n) = 5T(n/2) + n^2$ where $T(1) = 1$.
(D) $T(n) = 4T(n/2) + n^2$ where $T(1) = 1$.
(E) $T(n) = 3T(n/2) + n^2$ where $T(1) = 1$.

15.3. Graphs.

Let $G = (V, E)$ be an *unweighted*, undirected graph and let u and v be two vertices of G. Describe a linear time algorithm to find the number of shortest paths from u to v. Note that we only want the number of paths as there may be an exponential number of them. Give an example graph with an exponential number of (u, v)-paths.

15.4. Dynamic Programming.

The spring ROTC picnic at UIUC has fallen on a rainy day. The ranking officer decides to postpone the picnic and must notify everyone by phone. Here is the mechanism she uses to do this.

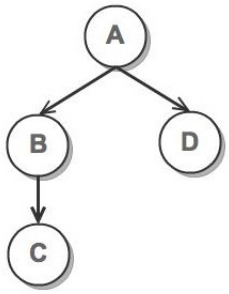
Each ROTC person on campus except the ranking officer reports to a unique *superior officer*. Thus the reporting hierarchy can be described by a tree T, rooted at the ranking officer, in which each other node v has a parent node u equal to his or her superior officer. Conversely, we will call v a *direct subordinate* of u. See the figure in which A is the ranking officer, B and D are direct subordinates of A, and C is the direct subordinate of B.

To notify everyone of the postponement, the ranking officer first calls each of her direct subordinates, one at a time. As soon as each subordinate gets the phone call, he or she must notify each of his or her direct subordinates, one at a time. The process continues this way until everyone has been notified. Note that each person in this process can only call direct subordinates on the phone; for example, in the figure, A would not be allowed to call C.

We can picture this process as being divided into *rounds*. In one round, each person who

has already learned of the postponement can call one of his or her direct subordinates on the phone. The number of rounds it takes for everyone to be notified depends on the sequence in which each person calls their direct subordinates. For example, in the figure, it will take only two rounds if A starts by calling B, but it will take three rounds if A starts by calling D.

Give an efficient algorithm that determines the minimum number of rounds needed for everyone to be notified.



15.5. Flow Reduction.

You're organizing the First Annual UIUC Computer Science 72-Hour Dance Exchange, to be held all day Friday, Saturday, and Sunday. Several 30-minute sets of music will be played during the event, and a large number of DJs have applied to perform. You need to hire DJs according to the following constraints:

- Exactly k sets of music must be played each day, and thus $3k$ sets altogether.
- Each set must be played by a single DJ in a consistent music genre (ambient, dubstep, trip-hop, J-pop, etc.).
- Each genre must be played at most once per day.
- Each candidate DJ has given you a list of genres they are willing to play.
- Each DJ can play at most three sets during the entire event.

Suppose there are n candidate DJs and g different musical genres available. Describe and analyze an efficient algorithm that either assigns a DJ and a genre to each of the $3k$ sets, or correctly reports that no such assignment is possible.

15.6. Pebbling Problem.

Pebbling is a solitaire game played on an undirected graph G, where each vertex has zero or more pebbles. A single pebbling move consists of removing two pebbles from a vertex v and adding one pebble to an arbitrary neighbor of v. (Obviously, the vertex v must have at least two pebbles before the move.) The PebbleDestruction problem asks, given a graph $G = (V, E)$ and a pebble count $p(v)$ for each vertex v, whether there is a sequence of pebbling moves that removes all but one pebble. Prove that PebbleDestruction is NP-complete.

15.7. Shortest Paths Reduction.

Consider a system of m linear inequalities over n variables $\{x_1, \cdots, x_n\}$. The kth inequality is in the form $x_{k_1} - x_{k_2} \leq t_k$ for $1 \leq k_1, k_2 \leq n$ and constant t_k (could be positive, zero or negative). The task is to present an algorithm that finds a solution of this system or indicates that the system has no solution.

(A) Build a graph G on vertex set $\{v_1, \cdots, v_n, s\}$. Put a directed edge from s to every v_i with weight 0. If the kth inequality is $x_{k_1} - x_{k_2} \leq t_k$, then put a directed edge from v_{k_2} to v_{k_1} in the graph with weight t_k. Let $d(s, v_i)$ be the length of the shortest path from s to v_i in the graph G. Are the values $d(s, v_i)$ guaranteed to exist? What algorithm could we use to compute all the values $d(s, v_i)$ if they exist?
Assuming that all $d(s, v_i)$ exist, prove that $x_i = d(s, v_i)$ for $1 \leq i \leq n$, is a solution to our linear system.

(B) Now assume that at least one of the values $d(s, v_i)$ does not exist. Prove that the linear system has no solution.

15.8. Certificates of positive answer.

For each of the following problems, describe what is the certificate that testifies that the answer to the given instance is positive, and how do you verify that the certificate is indeed correct.

(A) Min Cut.
Input: A flow network G and subset C of the edges.
Question: Is the set C is a minimum cut in G?

(B) Max Cut.
Input: An undirected graph G, and an integer k.
Question: Does the graph G contains an undirected cut with at least k edges in it?

(C) Independent Set.
Input: $\langle G, k \rangle$.
Question: Does G contains an independent set of size k?

(D) Clique.
Input: $\langle G, k \rangle$.
Question: Does G contains a clique of size k?

(E) Hamiltonian Cycle.
Input: A directed graph G.
Question: Does G contains a Hamiltonian cycle?

(F) SAT.
Input: A CNF formula F.
Question: Is there a satisfying assignment for F?

(G) Set Cover.
Input: A ground set $U = \{1, \ldots, m\}$, a family of subsets $\mathcal{F} = \{F_1, \ldots, F_n\}$, and a number k. Here $F_i \subseteq U$, for all i.
Question: Are there k subsets $F_{i_1}, \ldots, F_{i_k} \in \mathcal{F}$ such that $\bigcup_{j=1}^{k} F_{i_j} = U$.

(H) Partition.
Input: A set $X = \{x_1, \ldots, x_n\}$ of n positive numbers.

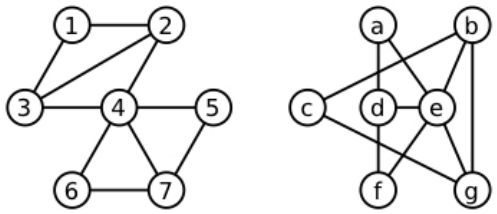
Question: Is there a subset $S \subseteq X$, such that $\sum_{x \in S} x = \sum_{x \in X \setminus S} x$?

(I) Graph Isomorphism.
Input: Two graphs $G = (V, E)$ and $G' = (V', E')$.
Question: Is G isomorphic to G'?
That is, is there a bijection $f : V \to V'$ such that for all $u, v \in V$, we have that $uv \in E$ if and only if $f(u)f(v) \in E'$.

All the problems mentioned above, except for Min Cut (which is solvable in polynomial time) and Graph Isomorphism are NP-Complete. For Graph Isomorphism it is currently open if it is NP-Complete (note, that Subgraph Isomorphism is NP-Complete).

15.9. Graph Isomorphism.

Two graphs are said to be ***isomorphic*** if one can be transformed into the other by relabeling the vertices. For example, the graphs shown below are isomorphic; the left graph can be transformed into the right graph by the relabeling $(1, 2, 3, 4, 5, 6, 7) \implies (c, g, b, e, a, f, d)$.



Two isomorphic graphs.

Consider the following related decision problems:

- Graph Isomorphism: Given two graphs G and H, determine whether G and H are isomorphic.
- Even Graph Isomorphism: Given two graphs G and H, such that every vertex in G and H has even degree, determine whether G and H are isomorphic.
- Subgraph Isomorphism: Given two graphs G and H, determine whether G is isomorphic to a subgraph of H.

(A) Describe a polynomial-time reduction from Graph Isomorphism to Even Graph Isomorphism.

(B) Describe a polynomial-time reduction from Graph Isomorphism to Subgraph Isomorphism.

(C) Prove that Subgraph Isomorphism is NP-Complete by reducing from Clique.

15.10. Self-Reductions.

In each case below, assume that you are given a black box which can answer the decision version of the indicated problem. Use a polynomial number of calls to the black box to construct the desired set.

(A) Subset sum: Given a multiset (elements can appear more than once) $X = x_1, \ldots, x_k$ of positive integers, and a positive integer S, does there exist a subset of X with sum exactly S?

(B) k-Color: Given a graph G, is there a proper k-coloring? In other words, can we assign one of the k colors to each node such that no node is adjacent to a node of the same color?

The same question for all the problems mentioned in (15.8).

15.11. 3-Dimensional Matching Problem.

We begin by discussing the 3-Dimensional Matching problem, which can be motivated as a harder version of the Bipartite Matching problem. The Bipartite Matching problem can be viewed in the following way: We are given two sets X and Y, each of size n, and a set P of pairs drawn from $X \times Y$. The question is: Does there exist a set of n pairs in P so that each element in $X \cup Y$ is contained in exactly one of these pairs?

Bipartite Matching is a problem we know how to solve in polynomial time, but things get much more complicated we move from ordered pairs to ordered triples. Consider the following 3-Dimensional Matching problem:

Given disjoint sets X, Y, Z, each of size n, and given a set $T \subseteq X \times Y \times Z$, does there exist a set of n triples in T so that each element of $X \cup Y \cup Z$ is contained in exactly one of these triples?

Such a triple is called a *perfect three-dimensional matching*. Show that 3-Dimensional Matching is NP-Complete by reducing from 3-SAT.