OLD CS 473: Fundamental Algorithms, Spring 2015

# More NP-Complete Problems

Lecture 24

April 23, 2015



## Recap

1. **NP**: languages that have polynomial time certifiers/verifiers
2. A language $L$ is **NP-Complete** iff
   - $L$ is in **NP**
   - for every $L'$ in **NP**, $L' \leq_P L$
3. $L$ is **NP-Hard** if for every $L'$ in **NP**, $L' \leq_P L$.
4. Cook-Levin theorem...

**Theorem (Cook-Levin)**

**Circuit-SAT** *and* **SAT** *are* **NP-Complete**.



## Recap contd

**Theorem (Cook-Levin)**

**Circuit-SAT** *and* **SAT** *are* **NP-Complete**.

1. Establish **NP-Complete**ness via reductions:
2. **SAT** $\leq_P$ **3SAT** and hence **3SAT** is **NP**-complete
3. **3SAT** $\leq_P$ **Independent Set** (which is in **NP**) and hence...
4. **Independent Set** is **NP-Complete**.
5. **Vertex Cover** is **NP-Complete**
6. **Clique** is **NP-Complete**.
7. **Set Cover** is **NP-Complete**.



## Today

Prove

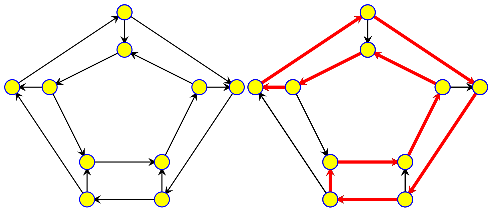
- **Hamiltonian Cycle** Problem is **NP-Complete**
- **3-Coloring** is **NP-Complete**

## Directed Hamiltonian Cycle

**Input** Given a directed graph $G = (V, E)$ with $n$ vertices

**Goal** Does $G$ have a Hamiltonian cycle?

- A Hamiltonian cycle is a cycle in the graph that visits every vertex in $G$ exactly once





## Directed Hamiltonian Cycle is NP-Complete

- Directed Hamiltonian Cycle is in $NP$
  - Certificate: Sequence of vertices
  - Certifier: Check if every vertex (except the first) appears exactly once, and that consecutive vertices are connected by a directed edge
- Hardness: We will show
  **3-SAT $\leq_P$ Directed Hamiltonian Cycle**



## Reduction

1. Given **3SAT** formula $\varphi$ create a graph $G_\varphi$ such that
   - $G_\varphi$ has a Hamiltonian cycle if and only if $\varphi$ is satisfiable
   - $G_\varphi$ should be constructible from $\varphi$ by a polynomial time algorithm $\mathcal{A}$
2. Notation: $\varphi$ has $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses $C_1, C_2, \ldots, C_m$.

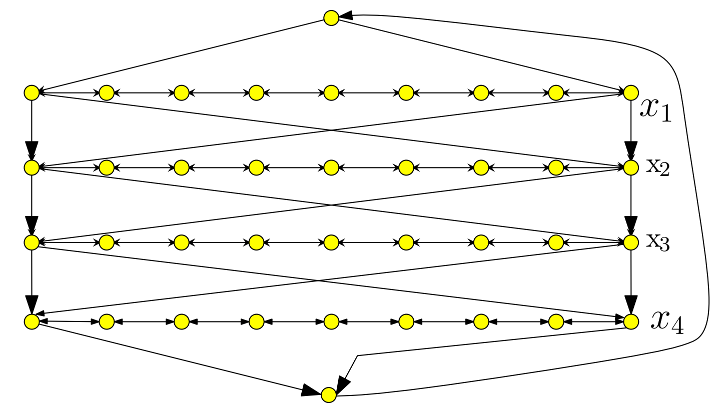


## Reduction: First Ideas

1. Viewing **SAT**: Assign values to $n$ variables, and each clauses has 3 ways in which it can be satisfied.
2. Construct graph with $2^n$ Hamiltonian cycles, where each cycle corresponds to some boolean assignment.
3. Then add more graph structure to encode constraints on assignments imposed by the clauses.

# The Reduction: Phase I

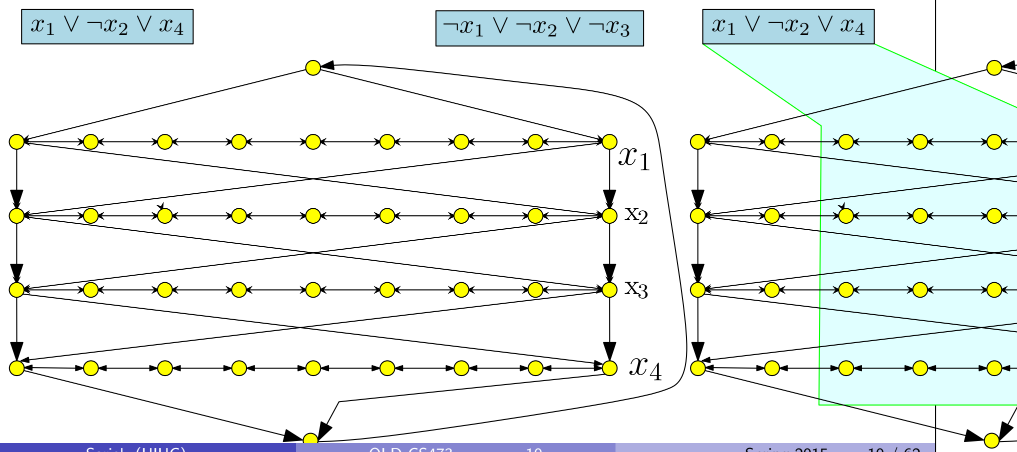
- Traverse path $i$ from left to right iff $x_i$ is set to true
- Each path has $3(m+1)$ nodes where $m$ is number of clauses in $\varphi$; nodes numbered from left to right ($1$ to $3m+3$)






# The Reduction: Phase II

- Add vertex $c_j$ for clause $C_j$. $c_j$ has edge *from* vertex $3j$ and *to* vertex $3j+1$ on path $i$ if $x_i$ appears in clause $C_j$, and has edge *from* vertex $3j+1$ and *to* vertex $3j$ if $\neg x_i$ appears in $C_j$.






# Correctness Proof

**Proposition**

*$\varphi$ has a satisfying assignment iff $G_\varphi$ has a Hamiltonian cycle.*

**Proof.**

$\Rightarrow$ Let $a$ be the satisfying assignment for $\varphi$. Define Hamiltonian cycle as follows

- If $a(x_i) = 1$ then traverse path $i$ from left to right
- If $a(x_i) = 0$ then traverse path $i$ from right to left
- For each clause, path of at least one variable is in the "right" direction to splice in the node corresponding to clause □

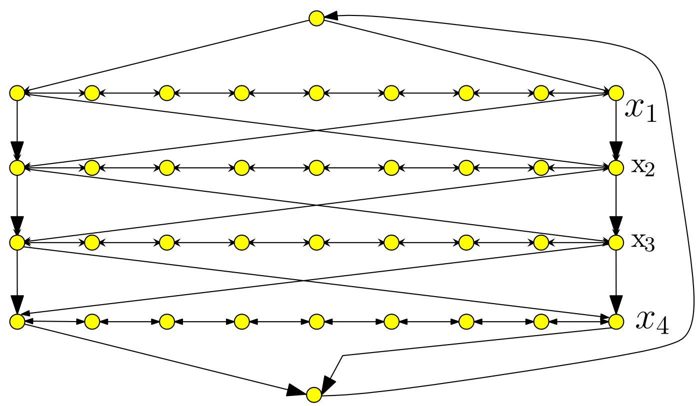


# Hamiltonian Cycle $\Rightarrow$ Satisfying assignment

Suppose $\Pi$ is a Hamiltonian cycle in $G_\varphi$

- If $\Pi$ enters $c_j$ (vertex for clause $C_j$) from vertex $3j$ on path $i$ then it must leave the clause vertex on edge to $3j+1$ on the *same path $i$*
  - If not, then only unvisited neighbor of $3j+1$ on path $i$ is $3j+2$
  - Thus, we don't have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle
- Similarly, if $\Pi$ enters $c_j$ from vertex $3j+1$ on path $i$ then it must leave the clause vertex $c_j$ on edge to $3j$ on path $i$

# Example






# Hamiltonian Cycle $\implies$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after $C_i$ are connected by an edge
- We can remove $c_j$ from cycle, and get Hamiltonian cycle in $G - c_j$
- Consider Hamiltonian cycle in $G - \{c_1, \ldots c_m\}$; it traverses each path in only one direction, which determines the truth assignment



# Hamiltonian Cycle

**Problem**

Input *Given undirected graph* $G = (V, E)$

Goal *Does* $G$ *have a Hamiltonian cycle? That is, is there a cycle that visits every vertex exactly one (except start and end vertex)?*



# NP-Completeness

**Theorem**

**Hamiltonian cycle** *problem for undirected graphs is* **NP-Complete***.*

**Proof.**

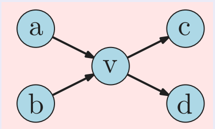
- The problem is in **NP**; proof left as exercise.
- Hardness proved by reducing Directed Hamiltonian Cycle to this problem □

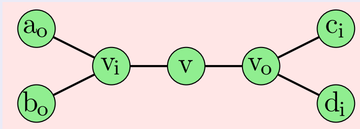
# Reduction Sketch

Goal: Given directed graph $G$, need to construct undirected graph $G'$ such that $G$ has Hamiltonian Path iff $G'$ has Hamiltonian path

## Reduction

- Replace each vertex $v$ by 3 vertices: $v_{in}$, $v$, and $v_{out}$
- A directed edge $(a, b)$ is replaced by edge $(a_{out}, b_{in})$






# Reduction: Wrapup

- The reduction is polynomial time (exercise)
- The reduction is correct (exercise)



# Graph Coloring

**Problem: Graph Coloring**

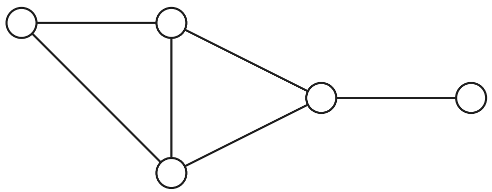
**Instance:** $G = (V, E)$: Undirected graph, integer $k$.
**Question:** Can the vertices of the graph be colored using $k$ colors so that vertices connected by an edge do not get the same color?

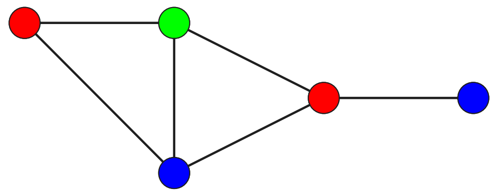


# Graph 3-Coloring

**Problem: 3 Coloring**

**Instance:** $G = (V, E)$: Undirected graph.
**Question:** Can the vertices of the graph be colored using **3** colors so that vertices connected by an edge do not get the same color?





# Graph Coloring

1. Observation: If $G$ is colored with $k$ colors then each color class (nodes of same color) form an independent set in $G$.
2. $G$ can be partitioned into $k$ independent sets iff $G$ is $k$-colorable.
3. Graph $2$-Coloring can be decided in polynomial time.
4. $G$ is $2$-colorable iff $G$ is bipartite!
5. There is a linear time algorithm to check if $G$ is bipartite using **BFS** (we saw this earlier).



# Graph Coloring and Register Allocation

## Register Allocation

Assign variables to (at most) $k$ registers such that variables needed at the same time are not assigned to the same register

## Interference Graph

Vertices are variables, and there is an edge between two vertices, if the two variables are "live" at the same time.

## Observations

- [Chaitin] Register allocation problem is equivalent to coloring the interference graph with $k$ colors
- Moreover, **3-COLOR** $\leq_P$ **k-Register Allocation**, for any $k \geq 3$



# Class Room Scheduling

1. Given $n$ classes and their meeting times, are $k$ rooms sufficient?
2. Reduce to Graph $k$-Coloring problem
3. Create graph $G$
   - a node $v_i$ for each class $i$
   - an edge between $v_i$ and $v_j$ if classes $i$ and $j$ *conflict*
4. Exercise: $G$ is $k$-colorable iff $k$ rooms are sufficient.



# Frequency Assignments in Cellular Networks

1. Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)
   - Breakup a frequency range $[a, b]$ into disjoint *bands* of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
   - Each cell phone tower (simplifying) gets one band
   - Constraint: nearby towers cannot be assigned same band, otherwise signals will interference
2. Problem: given $k$ bands and some region with $n$ towers, is there a way to assign the bands to avoid interference?
3. Can reduce to $k$-coloring by creating intereference/conflict graph on towers.

## 3-Coloring is NP-Complete

- 3-Coloring is in NP.
  - Certificate: for each node a color from $\{1, 2, 3\}$.
  - Certifier: Check if for each edge $(u, v)$, the color of $u$ is different from that of $v$.
- Hardness: We will show 3-SAT $\leq_P$ 3-Coloring.

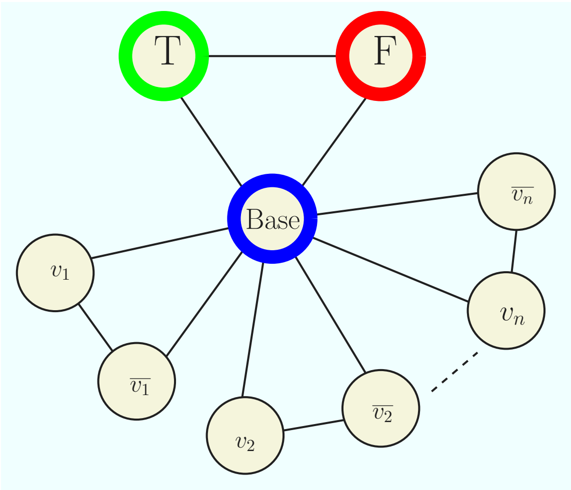


## Reduction Idea

1. $\varphi$: Given 3SAT formula (i.e., 3CNF formula).
2. $\varphi$: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph $G_\varphi$ s.t. $G_\varphi$ 3-colorable $\iff$ $\varphi$ satisfiable.
   - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of $G_\varphi$.
   - create triangle with node True, False, Base
   - for each variable $x_i$ two nodes $v_i$ and $\bar{v}_i$ connected in a triangle with common Base
   - If graph is 3-colored, either $v_i$ or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to $v_i$
   - Need to add constraints to ensure clauses are satisfied (next phase)



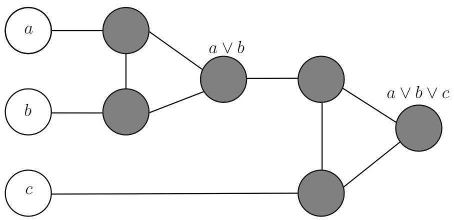
## Figure






## Clause Satisfiability Gadget

1. For each clause $C_j = (a \vee b \vee c)$, create a small gadget graph
   - gadget graph connects to nodes corresponding to $a, b, c$
   - needs to implement OR
2. OR-gadget-graph:



# OR-Gadget Graph

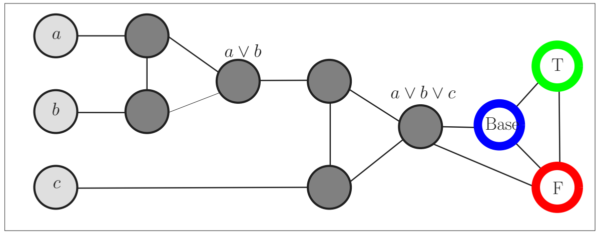
Property: if $a, b, c$ are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

Property: if one of $a, b, c$ is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.



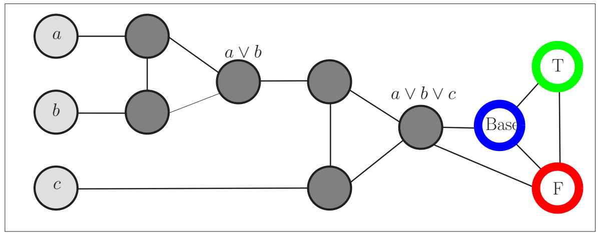
# Reduction

- create triangle with nodes True, False, Base
- for each variable $x_i$ two nodes $v_i$ and $\bar{v}_i$ connected in a triangle with common Base
- for each clause $C_j = (a \lor b \lor c)$, add OR-gadget graph with input nodes $a, b, c$ and connect output node of gadget to both False and Base






# Reduction

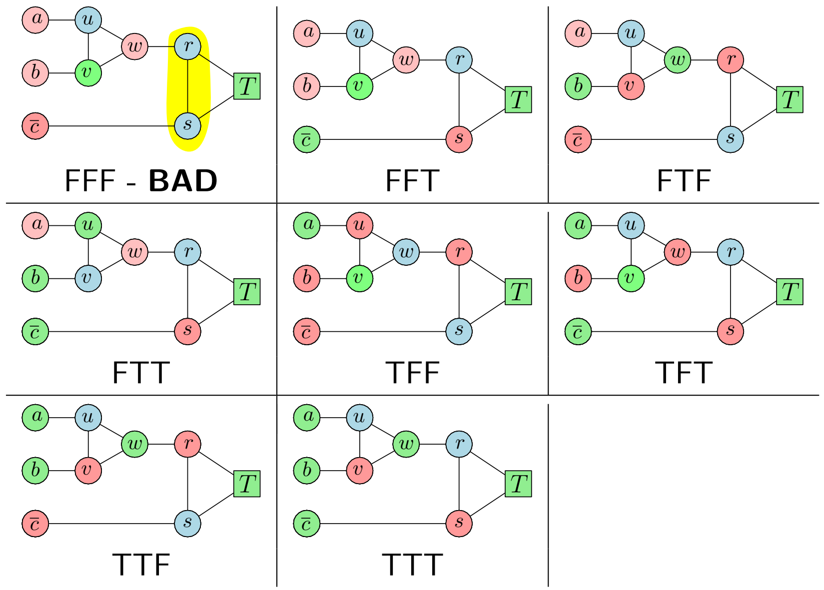



**Claim**

*No legal **3**-coloring of above graph (with coloring of nodes $T, F, B$ fixed) in which $a, b, c$ are colored False. If any of $a, b, c$ are colored True then there is a legal **3**-coloring of above graph.*



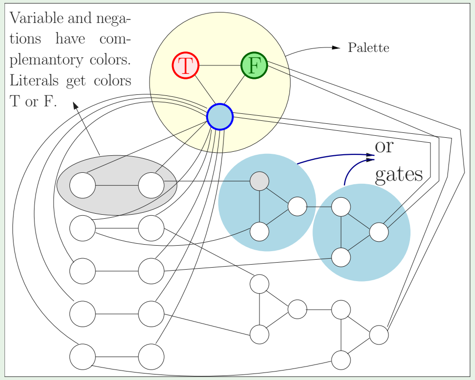
# 3 coloring of the clause gadget



# Reduction Outline

## Example

$\varphi = (u \lor \neg v \lor w) \land (v \lor x \lor \neg y)$






# Correctness of Reduction

$\varphi$ is satisfiable implies $G_\varphi$ is 3-colorable

- if $x_i$ is assigned True, color $v_i$ True and $\bar{v}_i$ False
- for each clause $C_j = (a \lor b \lor c)$ at least one of $a, b, c$ is colored True. OR-gadget for $C_j$ can be 3-colored such that output is True.

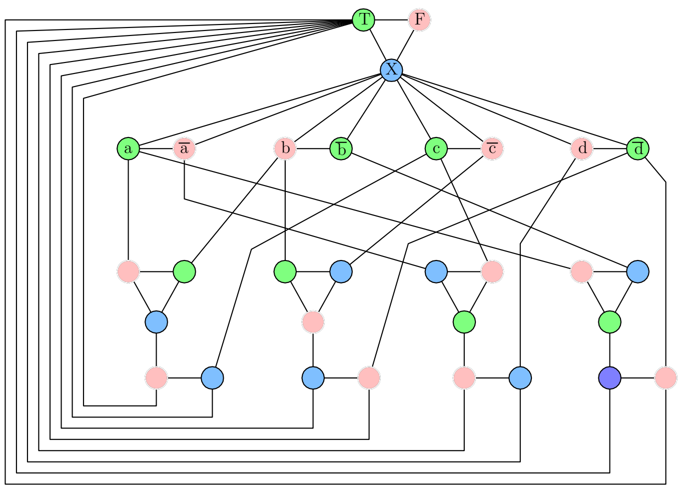
$G_\varphi$ is 3-colorable implies $\varphi$ is satisfiable

- if $v_i$ is colored True then set $x_i$ to be True, this is a legal truth assignment
- consider any clause $C_j = (a \lor b \lor c)$. it cannot be that all $a, b, c$ are False. If so, output of OR-gadget for $C_j$ has to be colored False but output is connected to Base and False!



# Graph generated in reduction...

... from 3SAT to 3COLOR






# Subset Sum

**Problem: Subset Sum**

**Instance:** $S$ - set of positive integers, $t$: - an integer number (Target)
**Question:** Is there a subset $X \subseteq S$ such that $\sum_{x \in X} x = t$?

## Claim

**Subset Sum** *is* **NP-Complete**.

## Vec Subset Sum

We will prove following problem is **NP-Complete**...

**Problem: Vec Subset Sum**

**Instance:** $S$ - set of $n$ vectors of dimension $k$, each vector has non-negative numbers for its coordinates, and a target vector $\overrightarrow{t}$.
**Question:** Is there a subset $X \subseteq S$ such that $\sum_{\overrightarrow{x} \in X} \overrightarrow{x} = \overrightarrow{t}$?

Reduction from **3SAT**.



## Vec Subset Sum

Handling a single clause

Think about vectors as being lines in a table.

**First gadget**

Selecting between two lines.

| Target | ?? | ?? | 01 | ??? |
|---|---|---|---|---|
| $a_1$ | ?? | ?? | 01 | ?? |
| $a_2$ | ?? | ?? | 01 | ?? |

Two rows for every variable $x$: selecting either $x = 0$ or $x = 1$.



## Handling a clause...

We will have a column for every clause...

| numbers | ... | $C \equiv a \vee b \vee \overline{c}$ | ... |
|---|---|---|---|
| $a$ | ... | 01 | ... |
| $\overline{a}$ | ... | 00 | ... |
| $b$ | ... | 01 | ... |
| $\overline{b}$ | ... | 00 | ... |
| $c$ | ... | 00 | ... |
| $\overline{c}$ | ... | 01 | ... |
| $C$ fix-up 1 | 000 | 07 | 000 |
| $C$ fix-up 2 | 000 | 08 | 000 |
| $C$ fix-up 3 | 000 | 09 | 000 |
| TARGET | | 10 | |



## 3SAT to Vec Subset Sum

| numbers | $a \vee \overline{a}$ | $b \vee \overline{b}$ | $c \vee \overline{c}$ | $d \vee \overline{d}$ | $D \equiv \overline{b} \vee c \vee \overline{d}$ | $C \equiv a \vee b \vee \overline{c}$ |
|---|---|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 | 00 | 01 |
| $\overline{a}$ | 1 | 0 | 0 | 0 | 00 | 00 |
| $b$ | 0 | 1 | 0 | 0 | 00 | 01 |
| $\overline{b}$ | 0 | 1 | 0 | 0 | 01 | 00 |
| $c$ | 0 | 0 | 1 | 0 | 01 | 00 |
| $\overline{c}$ | 0 | 0 | 1 | 0 | 00 | 01 |
| $d$ | 0 | 0 | 0 | 1 | 00 | 00 |
| $\overline{d}$ | 0 | 0 | 0 | 1 | 01 | 01 |
| $C$ fix-up 1 | 0 | 0 | 0 | 0 | 00 | 07 |
| $C$ fix-up 2 | 0 | 0 | 0 | 0 | 00 | 08 |
| $C$ fix-up 3 | 0 | 0 | 0 | 0 | 00 | 09 |
| $D$ fix-up 1 | 0 | 0 | 0 | 0 | 07 | 00 |
| $D$ fix-up 2 | 0 | 0 | 0 | 0 | 08 | 00 |
| $D$ fix-up 3 | 0 | 0 | 0 | 0 | 09 | 00 |
| TARGET | 1 | 1 | 1 | 1 | 10 | 10 |

# Vec Subset Sum to Subset Sum

| numbers |
| --- |
| 010000000001 |
| 010000000000 |
| 000100000001 |
| 000100000100 |
| 000001000100 |
| 000001000001 |
| 000000010000 |
| 000000010101 |
| 000000000007 |
| 000000000008 |
| 000000000009 |
| 000000000700 |
| 000000000800 |
| 000000000900 |
| 010101011010 |



# Other NP-Complete Problems

- 3-Dimensional Matching
- Subset Sum

Read book.



# Need to Know NP-Complete Problems

- **3SAT**.
- **Circuit-SAT**.
- **Independent Set**.
- **Vertex Cover**.
- **Clique**.
- **Set Cover**.
- **Hamiltonian Cycle** (in Directed/Undirected Graphs).
- **3Coloring**.
- **3-D Matching**.
- **Subset Sum**.



# Subset Sum and Knapsack

1. Subset Sum Problem: Given $n$ integers $a_1, a_2, \ldots, a_n$ and a target $B$, is there a subset of $S$ of $\{a_1, \ldots, a_n\}$ such that the numbers in $S$ add up *precisely* to $B$?
2. Subset Sum is **NP-Complete**— see book.
3. Knapsack: Given $n$ items with item $i$ having size $s_i$ and profit $p_i$, a knapsack of capacity $B$, and a target profit $P$, is there a subset $S$ of items that can be packed in the knapsack and the profit of $S$ is at least $P$?
4. Show Knapsack problem is **NP-Complete** via reduction from Subset Sum (exercise).

## Subset Sum and Knapsack

1. Subset Sum can be solved in $O(nB)$ time using dynamic programming (exercise).
2. Implies that problem is hard only when numbers $a_1, a_2, \ldots, a_n$ are exponentially large compared to $n$. That is, each $a_i$ requires polynomial in $n$ bits.
3. *Number problems* of the above type are said to be **weakly NPComplete**.



S. Even, A. Itai, and A. Shamir. On the complexity of timetable and multicommodity flow problems. *SIAM J. Comput.*, 5(4):691–703, 1976.