Chapter 21

Polynomial Time Reductions

OLD CS 473: Fundamental Algorithms, Spring 2015
April 14, 2015

21.0.1 Introduction to Reductions

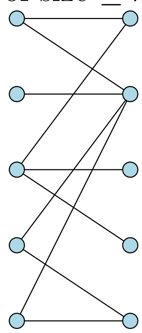
21.0.2 Overview

21.0.2.1 Reductions

(A) Reduction from Problem X to Problem Y (informally): having algorithm for Y, then have algorithm for Problem X.
(B) We use reductions to find algorithms to solve problems.
(C) We also use reductions to show that we **can't** find algorithms for some problems. (We say that these problems are **hard**.)
(D) Also, the right reductions might win you a million dollars!

21.0.2.2 Example 1: Bipartite Matching and Flows

How do we solve the **Bipartite Matching** Problem? Given a bipartite graph $G = (U \cup V, E)$ and number k, does G have a matching of size $\geq k$?



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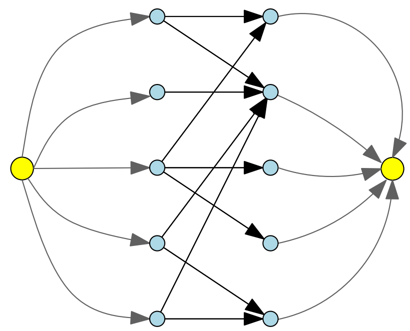

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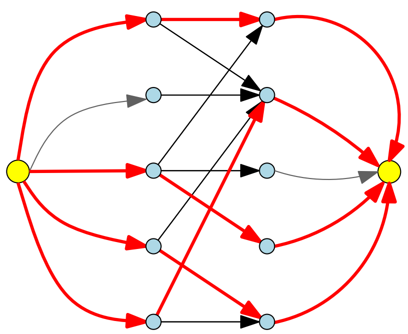

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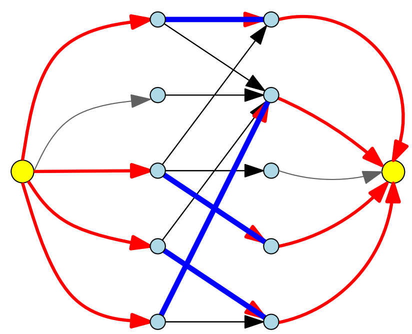


¡4-¿Solution Reduce it to **Max-Flow**. G has a matching of size $\geq k \iff$ there is a flow from s to t of value $\geq k$.

21.0.3 Definitions

21.0.3.1 Types of Problems

Decision, Search, and Optimization

(A) ***Decision problem***. Example: given n, **is** n prime?.

(B) ***Search problem***. Example: given n, **find** a factor of n if it exists.

(C) ***Optimization problem***. Example: find the **smallest** prime factor of n.

21.0.4 Optimization and Decision problems

21.0.4.1 For max flow...

(A) Max-flow as optimization problem:

Problem 21.0.1 (Max-Flow optimization version). *Given an instance G of network flow, find the maximum flow between s and t.*

(B) Max-flow as decision problem:

Problem 21.0.2 (Max-Flow decision version). *Given an instance* G *of network flow and a parameter* K*, is there a flow in* G*, from* s *to* t*, of value at least* K*?*

(C) While using reductions and comparing problems, we typically work with the decision versions. Decision problems have **Yes/No** answers. This makes them easy to work with.

21.0.4.2 Problems vs Instances

(A) A **problem** Π consists of an *infinite* collection of inputs $\{I_1, I_2, \ldots, \}$. Each input is referred to as an **instance**.
(B) The **size** of an instance I is the number of bits in its representation.
(C) For an instance I, $sol(I)$ is a set of **feasible solutions** to I.
(D) For optimization problems each solution $s \in sol(I)$ has an associated **value**.

21.0.4.3 Examples

(A) Instance **Bipartite Matching**: a bipartite graph, and integer k.
(B) Solution is "YES" if graph has matching size $\geq k$, else "NO".

(C) Instance **Max-Flow**: graph G with edge-capacities, two vertices s, t, and an integer k.
(D) Solution to instance is "YES" if there is a flow from s to t of value $\geq k$, else "NO".

(E) An algorithm for a decision Problem X?
(F) ***Decision algorithm***: Input an instance of X, and outputs either "YES" or "NO".

21.0.4.4 Encoding an instance into a string

(A) I; Instance of some problem.
(B) I can be fully and precisely described (say in a text file).
(C) Resulting text file is a binary string.
(D) $\implies$ Any input can be interpreted as a binary string S.
(E) ... Running time of algorithm: Function of length of S (i.e., n).

21.0.4.5 Decision Problems and Languages

(A) A finite **alphabet** Σ. Σ^* is set of all finite strings on Σ.
(B) A **language** L is simply a subset of Σ^*; a set of strings.
(C) Language $\equiv$ decision problem.
    (A) For any language $L \implies$ there is a decision problem Π_L.
    (B) For any decision problem $\Pi \implies$ an associated language L_Π.
(D) Given L, Π_L is the decision problem: Given $x \in \Sigma^*$, is $x \in L$? Each string in Σ^* is an instance of Π_L and L is the set of instances for which the answer is YES.
(E) Given Π the associated language is $L_\Pi = \left\{ I \;\middle|\; I \text{ is an instance of } \Pi \text{ for which answer is YES} \right\}$.
(F) Thus, decision problems and languages are used interchangeably.

21.0.4.6 Example

(A) The decision problem **Primality**, and the language

$$L = \left\{ \#p \;\middle|\; p \text{ is a prime number} \right\}.$$

Here $\#p$ is the string in base 10 representing p.

(B) **Bipartite** (is given graph is bipartite. The language is

$$L = \left\{ \mathcal{S}(\mathsf{G}) \;\middle|\; \mathsf{G} \text{ is a bipartite graph} \right\}.$$

Here $\mathcal{S}(\mathsf{G})$ is the string encoding the graph G.

21.0.4.7 Reductions, revised.

(A) For decision problems X, Y, a ***reduction from X to Y*** is:
   - (A) An algorithm ...
   - (B) Input: I_X, an instance of X.
   - (C) Output: I_Y an instance of Y.
   - (D) Such that:

$$\boxed{I_Y \text{ is YES instance of } Y} \iff \boxed{I_X \text{ is YES instance of } X}$$

(B) (Actually, this is only one type of reduction, but this is the one we'll use most often.)

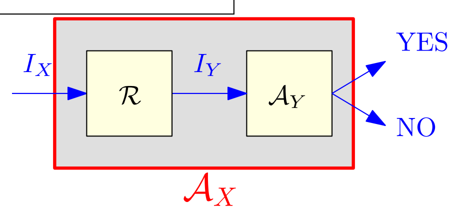
21.0.4.8 Using reductions to solve problems

(A) $\mathcal{R}$: Reduction $X \to Y$

(B) $\mathcal{A}_Y$: algorithm for Y:

(C) $\implies$ New algorithm for X:

```
A_X(I_X):
        // I_X: instance of X.
        I_Y ⇐ R(I_X)
        return A_Y(I_Y)
```



In particular, if $\mathcal{R}$ and $\mathcal{A}_Y$ are polynomial-time algorithms, $\mathcal{A}_X$ is also polynomial-time.

21.0.4.9 Comparing Problems

(A) Reductions allow us to formalize the notion of "Problem X is no harder to solve than Problem Y".

(B) If Problem X **reduces to** Problem Y (we write $X \leq Y$), then X cannot be harder to solve than Y.

(C) **Bipartite Matching** $\leq$ **Max-Flow**.
Therefore, **Bipartite Matching** cannot be harder than **Max-Flow**.

(D) Equivalently,
Max-Flow is **at least as hard as** **Bipartite Matching**.

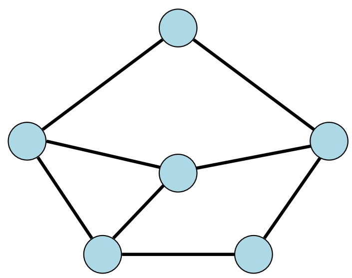
(E) More generally, if $X \leq Y$, we can say that X is no harder than Y, or Y is at least as hard as X.

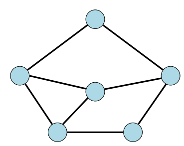
21.0.5 Examples of Reductions

21.0.6 Independent Set and Clique

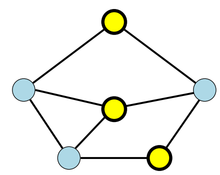
21.0.6.1 Independent Sets and Cliques

(A) Given a graph G.

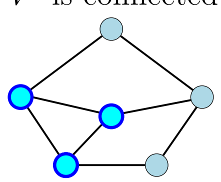




(B) A set of vertices V' is an ***independent set***: if no two vertices of V' are connected by an edge of G.



(C) ***clique***: *every* pair of vertices in V' is connected by an edge of G.



21.0.6.2 The Independent Set and Clique Problems

Problem: Independent Set

Instance: A graph G and an integer k. **Question:** Does G has an independent set of size $\geq k$?

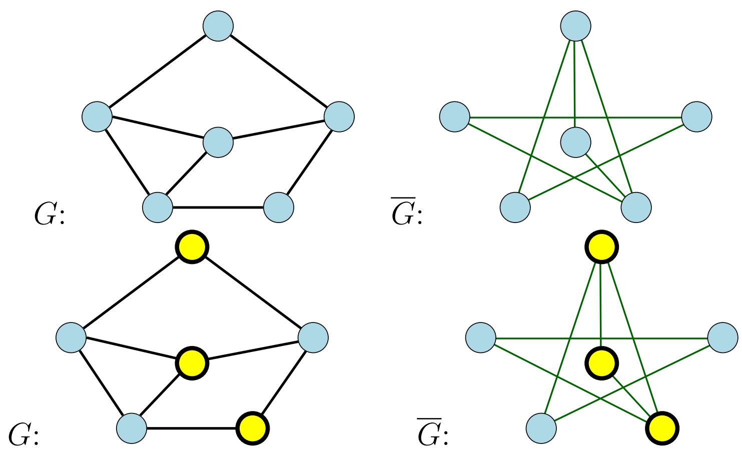
Problem: Clique

Instance: A graph G and an integer k. **Question:** Does G has a clique of size $\geq k$?

21.0.6.3 Recall

For decision problems X, Y, a reduction from X to Y is:
(A) An algorithm ...
(B) that takes I_X, an instance of X as input ...
(C) and returns I_Y, an instance of Y as output ...
(D) such that the solution (YES/NO) to I_Y is the same as the solution to I_X.

21.0.6.4 Reducing **Independent Set** to **Clique**



(A) An instance of **Independent Set** is a graph G and an integer k.
(B) Convert G to $\overline{G}$, in which (u, v) is an edge $\iff$ (u, v) is **not** an edge of G. ($\overline{G}$ is the *complement* of G.)
(C) ([)]$\overline{G}, k$: instance of **Clique**.

21.0.6.5 **Independent Set and Clique**

(A) **Independent Set** $\leq$ **Clique**.
What does this mean?
(B) If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
(C) **Clique** is *at least as hard as* **Independent Set**.
(D) Also... **Independent Set** is *at least as hard as* **Clique**.

21.0.7 NFAs/DFAs and Universality

21.0.7.1 DFAs and NFAs

(A) DFAs (Remember 373?) are determinstic automata that accept regular languages.
(B) NFAs are the same, except that non-deterministic.
(C) Every NFA can be converted to a DFA that accepts the same language using the **subset construction**.
(D) (How long does this take?)
(E) The smallest DFA equivalent to an NFA with n states may have $\approx 2^n$ states.

21.0.7.2 DFA Universality

(A) A DFA M is **universal** if it accepts every string.
(B) That is, $L(M) = \Sigma^*$, the set of all strings.
(C) DFA universality problem:

Problem 21.0.3 (DFA universality).
Input: *A DFA M.*
Goal: *Is M universal?*

(D) How do we solve **DFA Universality**?
(E) We check if M has *any* reachable non-final state.
(F) Alternatively, minimize M to obtain M' and see if M' has a single state which is an accepting state.

21.0.7.3 NFA Universality

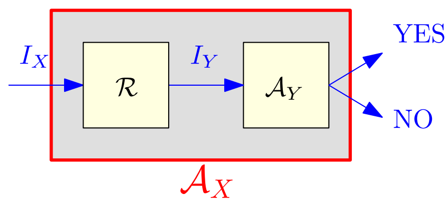
(A) An NFA N is **universal** if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.
(B) NFA universality problem:

Problem 21.0.4 (NFA universality).
Input: *A NFA M.*
Goal: *Is M universal?*

(C) How do we solve **NFA Universality**?
(D) Reduce it to **DFA Universality**...
(E) Given an NFA N, convert it to an equivalent DFA M, and use the **DFA Universality** Algorithm.
(F) The reduction takes **exponential time**!

21.0.7.4 Polynomial-time reductions

(A) An algorithm is ***efficient*** if it runs in polynomial-time.
(B) To find efficient algorithms for problems, we are only interested in **polynomial-time** reductions. Reductions that take longer are not useful.
(C) If we have a polynomial-time reduction from problem X to problem Y (we write $X \leq_P Y$), and a poly-time algorithm $\mathcal{A}_Y$ for Y, we have a polynomial-time/efficient algorithm for X.



21.0.7.5 Polynomial-time Reduction

(A) A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:

    (A) given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
    (B) $\mathcal{A}$ runs in time polynomial in $|I_X|$.
    (C) Answer to I_X YES $\iff$ answer to I_Y is YES.
(B) Polynomial transitivity:

Proposition 21.0.5. *If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.*

(C) Such a reduction is a ***Karp reduction***. Most reductions we will need are Karp reductions.

21.0.7.6 Polynomial-time reductions and hardness

(A) For decision problems X and Y, if $X \leq_P Y$, and Y has an efficient algorithm, X has an efficient algorithm.
(B) If you believe that **Independent Set** does not have an efficient algorithm, why should you believe the same of **Clique**?
(C) Because we showed **Independent Set** $\leq_P$ **Clique**. If **Clique** had an efficient algorithm, so would **Independent Set**!
(D) If $X \leq_P Y$ and X does not have an efficient algorithm, Y cannot have an efficient algorithm!

21.0.7.7 Polynomial-time reductions and instance sizes

Proposition 21.0.6. *Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.*

Proof: $\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.

I_Y is the output of $\mathcal{R}$ on input I_X.

$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. ■

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

21.0.7.8 Polynomial-time Reduction

A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:
(A) Given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y.
(B) $\mathcal{A}$ runs in time polynomial in $|I_X|$. This implies that $|I_Y|$ (size of I_Y) is polynomial in $|I_X|$.
(C) Answer to I_X YES *iff* answer to I_Y is YES.

Proposition 21.0.7. *If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.*

Such a reduction is called a Karp reduction. Most reductions we will need are Karp reductions

21.0.7.9 Transitivity of Reductions

(A) Reductions are transitive:

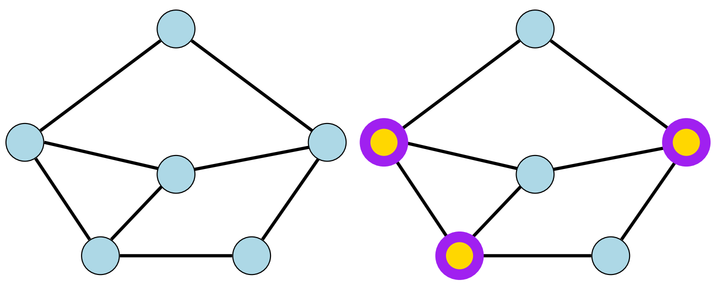
Proposition 21.0.8. *$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.*

(B) **Note:** $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.
(C) To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y.
(D) In other words show that an algorithm for Y implies an algorithm for X.

21.0.8 Independent Set and Vertex Cover

21.0.8.1 Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:
(A) A ***vertex cover*** if every $e \in E$ has at least one endpoint in S.



21.0.8.2 The **Vertex Cover** Problem

Problem 21.0.9 (Vertex Cover).
Input: *A graph G and integer k.*
Goal: *Is there a vertex cover of size $\leq k$ in G?*

Can we relate **Independent Set** and **Vertex Cover**?

21.0.9 Relationship between...

21.0.9.1 Vertex Cover and Independent Set

Proposition 21.0.10. *Let $G = (V, E)$ be a graph. S is an independent set if and only if $V \setminus S$ is a vertex cover.*

Proof:
($\Rightarrow$) Let S be an independent set
  (A) Consider any edge $uv \in E$.
  (B) Since S is an independent set, either $u \notin S$ or $v \notin S$.
  (C) Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.

(D) $V \setminus S$ is a vertex cover.

($\Leftarrow$) Let $V \setminus S$ be some vertex cover:

(A) Consider $u, v \in S$

(B) uv is not an edge of G, as otherwise $V \setminus S$ does not cover uv.

(C) $\implies$ S is thus an independent set.

■

21.0.9.2 Independent Set $\leq_P$ Vertex Cover

(A) G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.

(B) G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$

(C) (G, k) is an instance of **Independent Set** , and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.

(D) Therefore, **Independent Set** $\leq_P$ **Vertex Cover**. Also **Vertex Cover** $\leq_P$ **Independent Set**.

21.0.10 Vertex Cover and Set Cover

21.0.10.1 A problem of Languages

(A) Suppose you work for the United Nations. Let U be the set of all **languages** spoken by people across the world. The United Nations also has a set of **translators**, all of whom speak English, and some other languages from U.

(B) Due to budget cuts, you can only afford to keep k translators on your payroll. Can you do this, while still ensuring that there is someone who speaks every language in U?

(C) More General problem: Find/Hire a small group of people who can accomplish a large number of tasks.

21.0.10.2 The Set Cover Problem

Problem 21.0.11 (Set Cover).

Input: *Given a set U of n elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U, and an integer k.*

Goal: *Is there a collection of at most k of these sets S_i whose union is equal to U?*

Example 21.0.12. *¡2-¿Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ with*

$$\begin{array}{ll} S_1 = \{3,7\} & ¡3- > S_2 = \{3,4,5\} \\ S_3 = \{1\} & S_4 = \{2,4\} \\ S_5 = \{5\} & ¡3- > S_6 = \{1,2,6,7\} \end{array}$$

$\{S_2, S_6\}$ *is a set cover*

21.0.10.3 Vertex Cover $\leq_P$ Set Cover

Given graph $G = (V, E)$ and integer k as instance of **Vertex Cover**, construct an instance of **Set Cover** as follows:

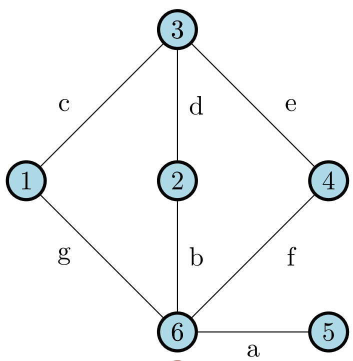
(A) Number k for the **Set Cover** instance is the same as the number k given for the **Vertex Cover** instance.

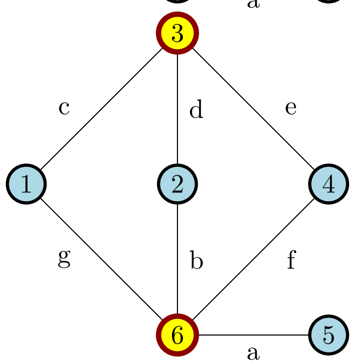
(B) $U = E$.

(C) We will have one set corresponding to each vertex; $S_v = \{e \mid e \text{ is incident on } v\}$.

Observe that G has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

21.0.10.4 Vertex Cover $\leq_P$ Set Cover: Example





Let $U = \{a, b, c, d, e, f, g\}$, $k = 2$ with

$S_1 = \{c, g\}$	$S_2 = \{b, d\}$
¡3$-> S_3 = \{c, d, e\}$	$S_4 = \{e, f\}$
$S_5 = \{a\}$	¡3$-> S_6 = \{a, b, f, g\}$

$\{S_3, S_6\}$ is a set cover

$\{3, 6\}$ is a vertex cover

21.0.10.5 Proving Reductions

To prove that $X \leq_P Y$ you need to give an algorithm $\mathcal{A}$ that:

(A) Transforms an instance I_X of X into an instance I_Y of Y.

(B) Satisfies the property that answer to I_X is YES iff I_Y is YES.

  (A) typical easy direction to prove: answer to I_Y is YES if answer to I_X is YES

  (B) **typical difficult direction to prove**: answer to I_X is YES if answer to I_Y is YES (equivalently answer to I_X is NO if answer to I_Y is NO).

(C) Runs in *polynomial* time.

21.0.10.6 Example of incorrect reduction proof

Try proving **Matching** $\leq_P$ **Bipartite Matching** via following reduction:

(A) Given graph $G = (V, E)$ obtain a bipartite graph $G' = (V', E')$ as follows.

  (A) Let $V_1 = \{u_1 \mid u \in V\}$ and $V_2 = \{u_2 \mid u \in V\}$. We set $V' = V_1 \cup V_2$ (that is, we make two copies of V)

  (B) $E' = \left\{ u_1 v_2 \,\middle|\, u \neq v \text{ and } uv \in E \right\}$

(B) Given G and integer k the reduction outputs G' and k.

21.0.10.7 Example

21.0.10.8 "Proof"

Claim 21.0.13. *Reduction is a poly-time algorithm. If G has a matching of size k then G' has a matching of size k.*

Proof: Exercise. ■

Claim 21.0.14. *If G' has a matching of size k then G has a matching of size k.*

Incorrect! Why? Vertex $u \in V$ has two copies u_1 and u_2 in G'. A matching in G' may use both copies!

21.0.10.9 Summary

(A) We looked at **polynomial-time reductions**.

(B) Using polynomial-time reductions

  (A) If $X \leq_P Y$, and we have an efficient algorithm for Y, we have an efficient algorithm for X.

  (B) If $X \leq_P Y$, and there is no efficient algorithm for X, there is no efficient algorithm for Y.

(C) We looked at some examples of reductions between **Independent Set**, **Clique**, **Vertex Cover**, and **Set Cover**.