Polynomial Time Reductions

Lecture 21

April 14, 2015

21.1: Introduction to Reductions

21.2: Overview

Reductions

1. Reduction from Problem X to Problem Y (informally): having algorithm for Y, then have algorithm for Problem X.
2. We use reductions to find algorithms to solve problems.
3. We also use reductions to show that we can't find algorithms for some problems. (We say that these problems are hard.)
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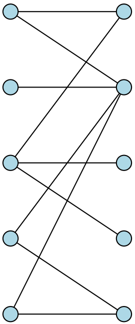
Example 1: Bipartite Matching and Flows

How do we solve the Bipartite Matching Problem?

Given a bipartite graph $G = (U \cup V, E)$ and number k, does G have a matching of size $\geq k$?



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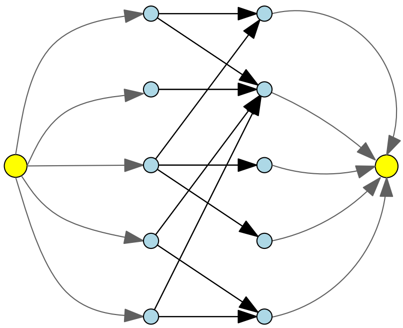

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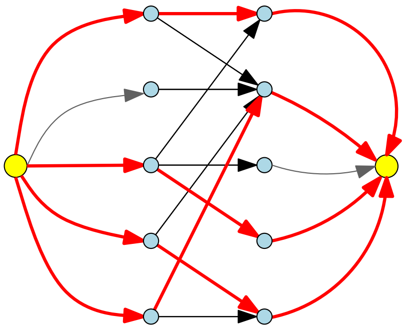

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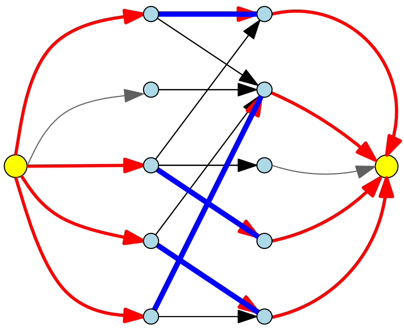

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21.3: Definitions

Types of Problems

Decision, Search, and Optimization

1. **Decision problem**. Example: given n, is n prime?.
2. **Search problem**. Example: given n, find a factor of n if it exists.
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Optimization and Decision problems

For max flow...

1. Max-flow as optimization problem:

Problem (Max-Flow optimization version)

Given an instance G of network flow, find the maximum flow between s and t.

2. Max-flow as decision problem:

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Given an instance G of network flow and a parameter K, is there a flow in G, from s to t, of value at least K?

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Problems vs Instances

1. A problem Π consists of an **infinite** collection of inputs $\{I_1, I_2, \ldots,\}$. Each input is referred to as an instance.
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3. For an instance I, $sol(I)$ is a set of feasible solutions to I.
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Decision Problems and Languages

1. A finite alphabet Σ. Σ^* is set of all finite strings on Σ.
2. A language L is simply a subset of Σ^*; a set of strings.
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Example

1. The decision problem **Primality**, and the language

$$L = \left\{ \#p \;\middle|\; p \text{ is a prime number} \right\}.$$

Here $\#p$ is the string in base $\mathbf{10}$ representing p.

2. **Bipartite** (is given graph is bipartite. The language is

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Reductions, revised.

1. For decision problems X, Y, a **reduction from X to Y** is:
   1. An algorithm ...
   2. Input: I_X, an instance of X.
   3. Output: I_Y an instance of Y.
   4. Such that:

$$\boxed{I_Y \text{ is YES instance of } Y} \iff \boxed{I_X \text{ is YES instance of } X}$$

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Using reductions to solve problems

1. $\mathcal{R}$: Reduction $X \rightarrow Y$
2. $\mathcal{A}_Y$: algorithm for Y:
3. $\implies$ New algorithm for X:

$\mathcal{A}_X(I_X)$:
    // I_X: instance of X.
    $I_Y \Leftarrow \mathcal{R}(I_X)$
    return $\mathcal{A}_Y(I_Y)$

If $\mathcal{R}$ and $\mathcal{A}_Y$ polynomial-time $\implies$ $\mathcal{A}_X$ polynomial-time.

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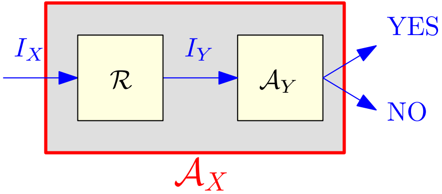

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Comparing Problems

1. "Problem X is no harder to solve than Problem Y".
2. If Problem X reduces to Problem Y (we write $X \leq Y$), then X cannot be harder to solve than Y.
3. **Bipartite Matching** $\leq$ **Max-Flow**.
   Bipartite Matching cannot be harder than **Max-Flow**.
4. Equivalently,
   Max-Flow is at least as hard as **Bipartite Matching**.
5. $X \leq Y$:
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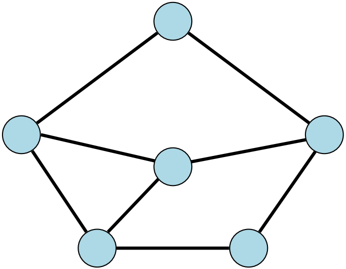
21.4: Examples of Reductions

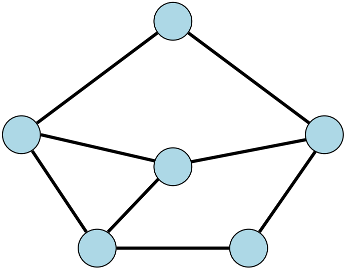
21.4.1: Independent Set and Clique

Independent Sets and Cliques



1. Given a graph G.
2. A set of vertices V' is an **independent set**: no two vertices of V' connected by an edge.
3. **clique**: *every* pair of vertices in V' is connected by an edge of G.

Independent Sets and Cliques



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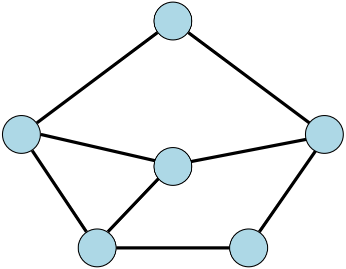

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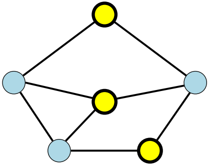

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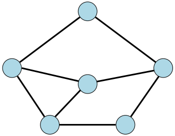
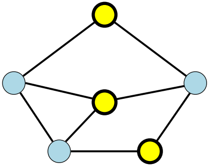
1. Given a graph G. 

2. A set of vertices V' is an **independent set**: no two vertices of V' connected by an edge. 

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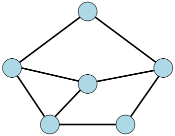
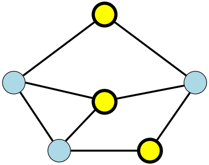
1. Given a graph G. 
2. A set of vertices V' is an **independent set**: no two vertices of V' connected by an edge. 
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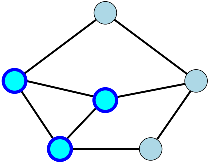

The Independent Set and Clique Problems

Problem: Independent Set

Instance: A graph G and an integer k.
Question: Does G has an independent set of size $\geq k$?

Problem: Clique

Instance: A graph G and an integer k.
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Recall

For decision problems X, Y, a reduction from X to Y is:

1. An algorithm ...
2. that takes I_X, an instance of X as input ...
3. and returns I_Y, an instance of Y as output ...
4. such that the solution (YES/NO) to I_Y is the same as the solution to I_X.

Reducing **Independent Set** to **Clique**

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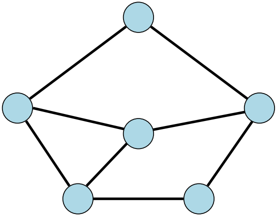

1. An instance of **Independent Set** is a graph G and an integer k.
2. Convert G to $\overline{G}$, in which (u, v) is an edge $\iff$ (u, v) is not an edge of G. ($\overline{G}$ is the *complement* of G.)
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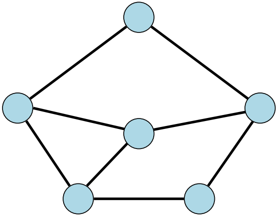

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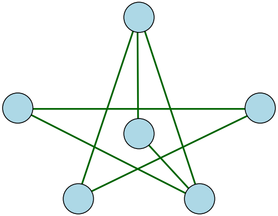

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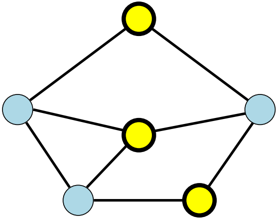

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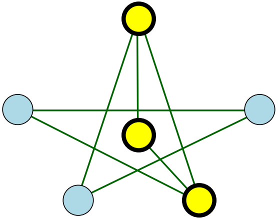

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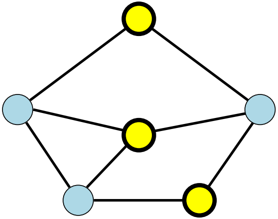

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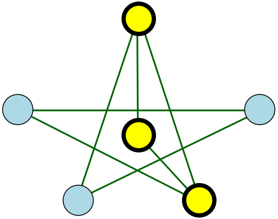

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21.4.2: NFAs/DFAs and Universality

DFAs and NFAs

1. DFAs (Remember 373?) are determinstic automata that accept regular languages.
2. NFAs are the same, except that non-deterministic.
3. Every NFA can be converted to a DFA that accepts the same language using the subset construction.
4. (How long does this take?)
5. The smallest DFA equivalent to an NFA with n states may have $\approx 2^n$ states.

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DFA Universality

1. A DFA M is universal if it accepts every string.
2. That is, $L(M) = \Sigma^*$, the set of all strings.
3. DFA universality problem:

Problem (DFA universality)

Input: *A DFA M.*
Goal: *Is M universal?*

4. How do we solve **DFA Universality**?
5. We check if M has *any* reachable non-final state.
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1. An algorithm is **efficient** if it runs in polynomial-time.
2. To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.
3. If we have a polynomial-time reduction from problem X to problem Y (we write $X \leq_P Y$), and a poly-time algorithm $\mathcal{A}_Y$ for Y, we have a polynomial-time/efficient algorithm for X.

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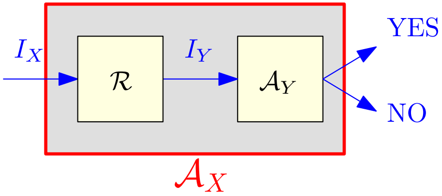

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   1. given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
   2. $\mathcal{A}$ runs in time polynomial in $|I_X|$.
   3. Answer to I_X YES $\iff$ answer to I_Y is YES.
2. Polynomial transitivity:

Proposition

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

3. Such a reduction is a **Karp reduction**. Most reductions we will need are Karp reductions.

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Polynomial-time reductions and hardness

1. For decision problems X and Y, if $X \leq_P Y$, and Y has an efficient algorithm, X has an efficient algorithm.
2. If you believe that **Independent Set** does not have an efficient algorithm, why should you believe the same of **Clique**?
3. Because we showed **Independent Set** $\leq_P$ **Clique**. If **Clique** had an efficient algorithm, so would **Independent Set**!
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Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.
I_Y is the output of $\mathcal{R}$ on input I_X.
$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \le p(|I_X|)$. $\square$

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

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Such a reduction is called a Karp reduction. Most reductions we will need are Karp reductions

Transitivity of Reductions

1. Reductions are transitive:

Proposition

$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.

2. Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.
3. To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y.
4. In other words show that an algorithm for Y implies an algorithm for X.

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$X \leq_P Y$ *and* $Y \leq_P Z$ *implies that* $X \leq_P Z$.

2. Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.
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1. Reductions are transitive:

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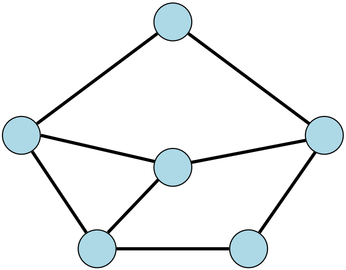
21.5: Independent Set and Vertex Cover

Vertex Cover

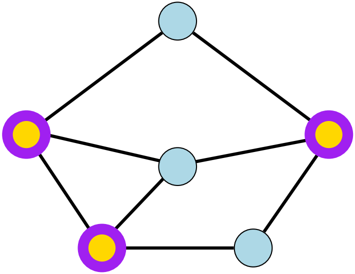
Given a graph $G = (V, E)$, a set of vertices S is:



Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

1. A **vertex cover** if every $e \in E$ has at least one endpoint in S.



The Vertex Cover Problem

Problem (Vertex Cover)

Input: *A graph G and integer k.*
Goal: *Is there a vertex cover of size $\leq k$ in G?*

Can we relate **Independent Set** and **Vertex Cover**?

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Can we relate **Independent Set** and **Vertex Cover**?

Proposition

Let $G = (V, E)$ be a graph. S is an independent set if and only if $V \setminus S$ is a vertex cover.

Proof.

($\Rightarrow$) Let S be an independent set

1. Consider any edge $uv \in E$.
2. Since S is an independent set, either $u \notin S$ or $v \notin S$.
3. Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
4. $V \setminus S$ is a vertex cover.

($\Leftarrow$) Let $V \setminus S$ be some vertex cover:

1. Consider $u, v \in S$
2. uv is not an edge of G, as otherwise $V \setminus S$ does not cover uv.
3. $\implies$ S is thus an independent set. □

Relationship between...

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Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$
3. (G, k) is an instance of **Independent Set** , and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
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21.6: Vertex Cover and Set Cover

A problem of Languages

1. Suppose you work for the United Nations. Let U be the set of all languages spoken by people across the world. The United Nations also has a set of translators, all of whom speak English, and some other languages from U.
2. Due to budget cuts, you can only afford to keep k translators on your payroll. Can you do this, while still ensuring that there is someone who speaks every language in U?
3. More General problem: Find/Hire a small group of people who can accomplish a large number of tasks.

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The Set Cover Problem

Problem (Set Cover)

Input: *Given a set U of n elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U, and an integer k.*

Goal: *Is there a collection of at most k of these sets S_i whose union is equal to U?*

Example

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ with

$$\begin{array}{ll} S_1 = \{3, 7\} & S_2 = \{3, 4, 5\} \\ S_3 = \{1\} & S_4 = \{2, 4\} \\ S_5 = \{5\} & S_6 = \{1, 2, 6, 7\} \end{array}$$

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Given graph $G = (V, E)$ and integer k as instance of **Vertex Cover**, construct an instance of **Set Cover** as follows:

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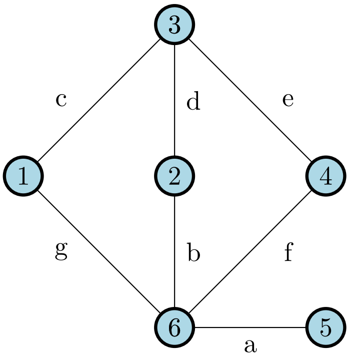
Observe that G has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

Vertex Cover $\leq_P$ Set Cover: Example



Let $U = \{a, b, c, d, e, f, g\}$, $k = 2$ with

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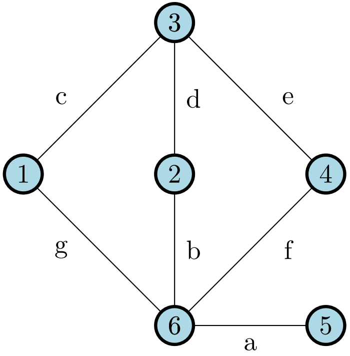

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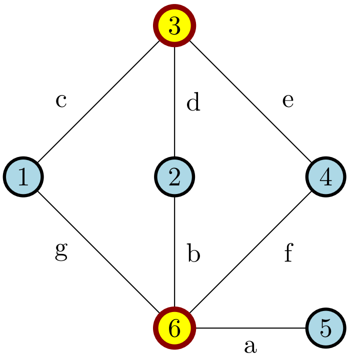

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Proving Reductions

To prove that $X \leq_P Y$ you need to give an algorithm $\mathcal{A}$ that:

1. Transforms an instance I_X of X into an instance I_Y of Y.
2. Satisfies the property that answer to I_X is YES iff I_Y is YES.
   1. typical easy direction to prove: answer to I_Y is YES if answer to I_X is YES
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Example of incorrect reduction proof

Try proving **Matching** $\leq_P$ **Bipartite Matching** via following reduction:

1. Given graph $G = (V, E)$ obtain a bipartite graph $G' = (V', E')$ as follows.
   1. Let $V_1 = \{u_1 \mid u \in V\}$ and $V_2 = \{u_2 \mid u \in V\}$. We set $V' = V_1 \cup V_2$ (that is, we make two copies of V)
   2. $E' = \left\{ u_1 v_2 \;\middle|\; u \neq v \text{ and } uv \in E \right\}$
2. Given G and integer k the reduction outputs G' and k.

Example

"Proof"

Claim

Reduction is a poly-time algorithm. If G has a matching of size k then G' has a matching of size k.

Proof.

Exercise. □

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If G' has a matching of size k then G has a matching of size k.

Incorrect! Why? Vertex $u \in V$ has two copies u_1 and u_2 in G'. A matching in G' may use both copies!

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Summary

1. We looked at polynomial-time reductions.
2. Using polynomial-time reductions
   1. If $X \leq_P Y$, and we have an efficient algorithm for Y, we have an efficient algorithm for X.
   2. If $X \leq_P Y$, and there is no efficient algorithm for X, there is no efficient algorithm for Y.
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