Network Flow Algorithms

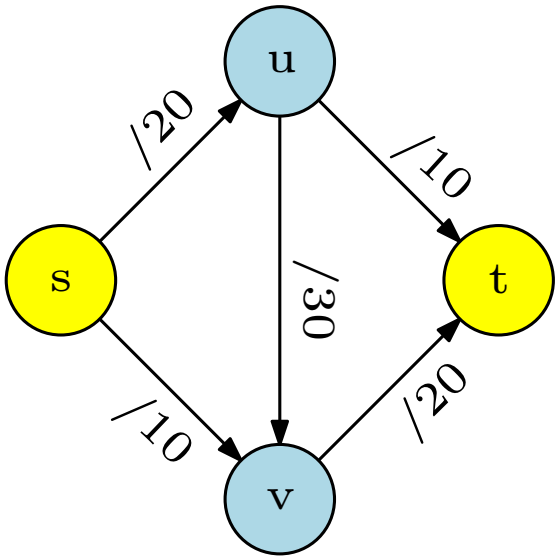
Lecture 18

March 31, 2015

Part I

Algorithm(s) for Maximum Flow

Greedy Approach



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2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$.
3. **Augment** flow along this path.
4. Repeat augmentation for as long as possible.

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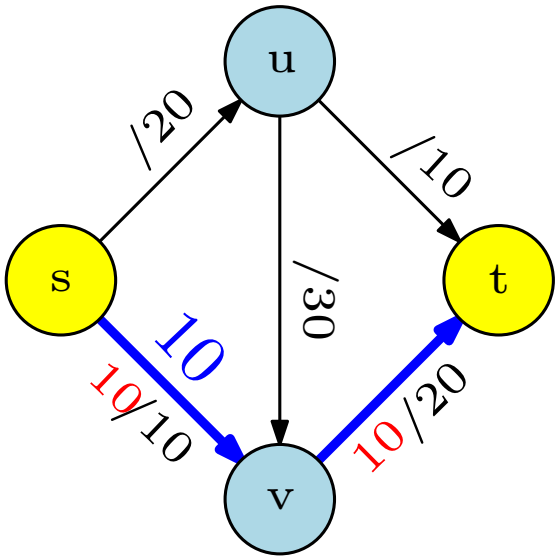

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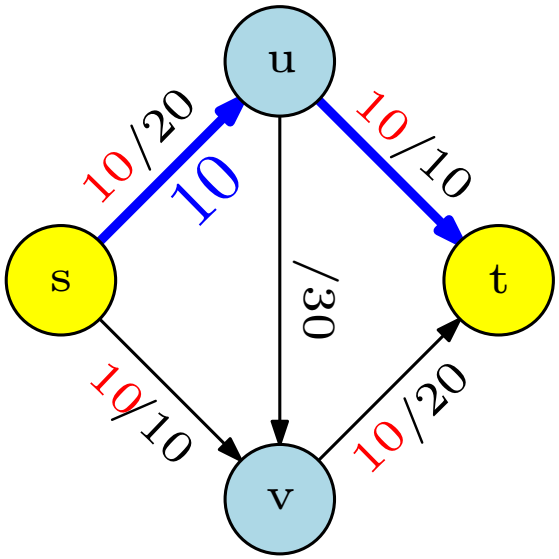

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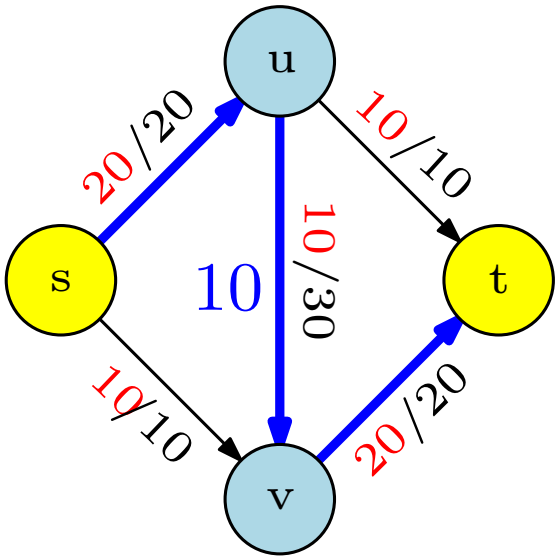

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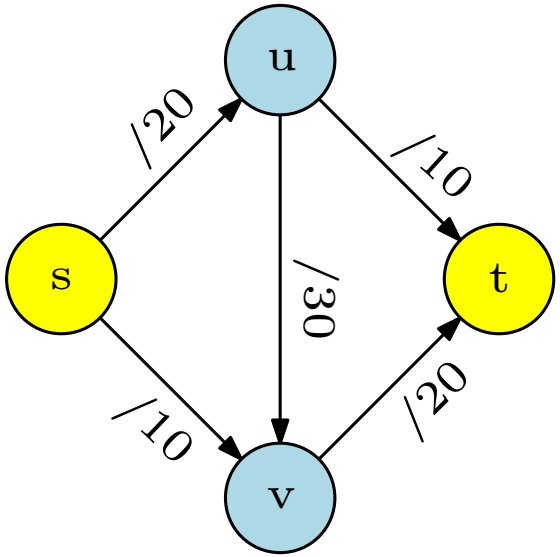

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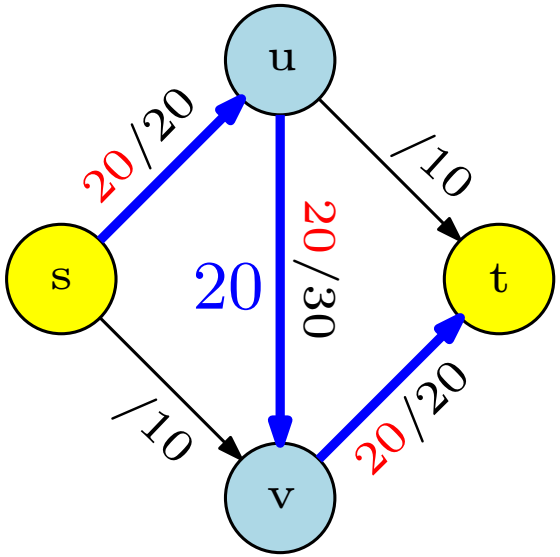

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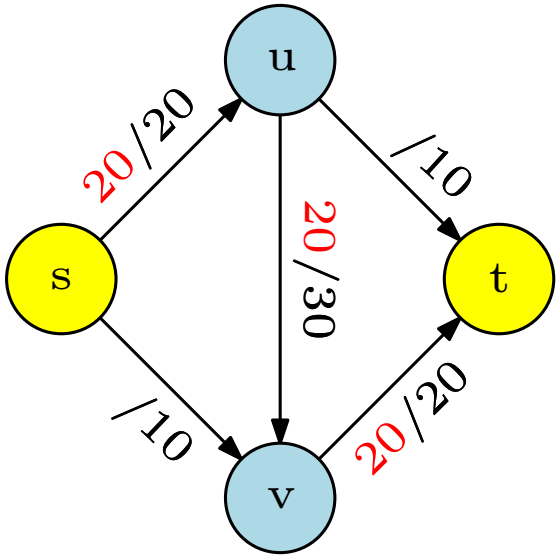

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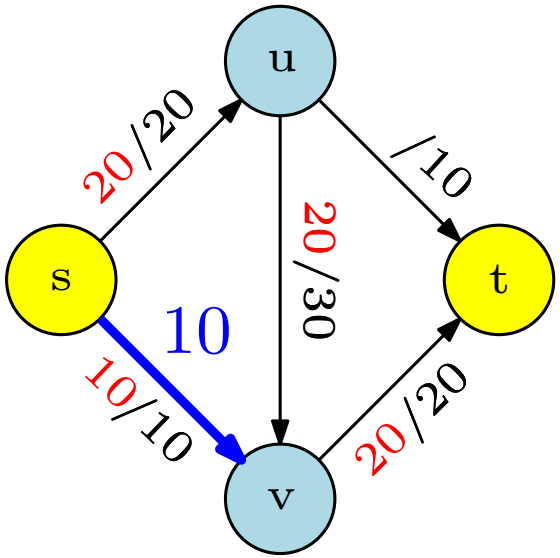

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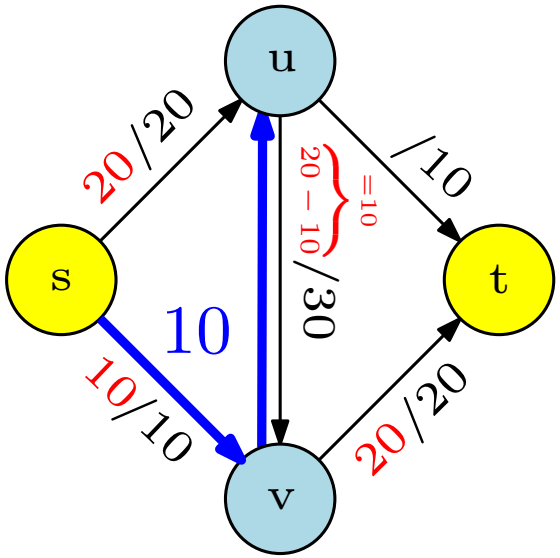

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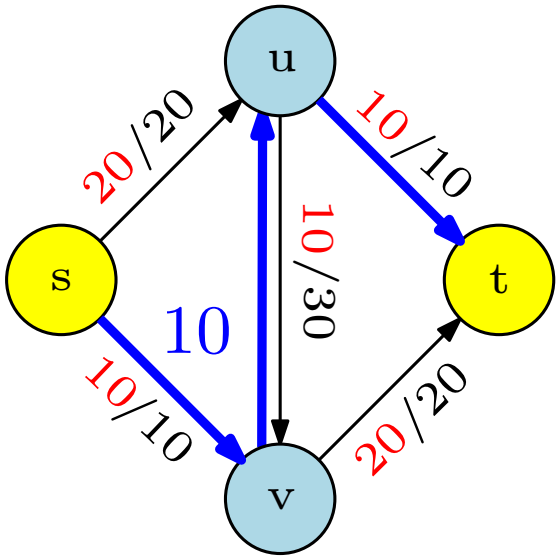

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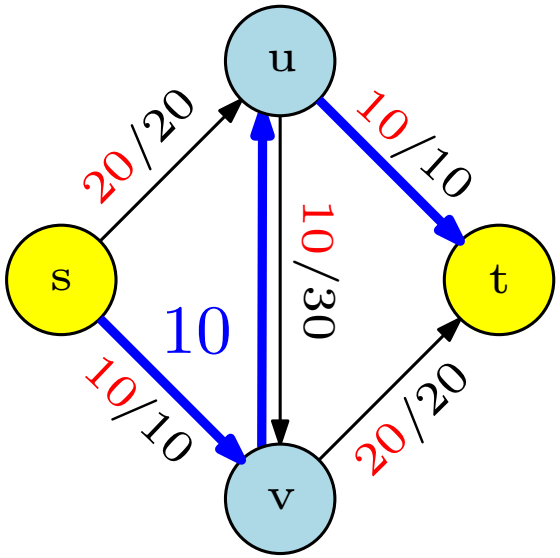

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Residual Graph

The "leftover" graph

Definition

For a network $G = (V, E)$ and flow f, the **residual graph** $G_f = (V', E')$ of G with respect to f is

1. $V' = V$,
2. **Forward Edges**: For each edge $e \in E$ with $f(e) < c(e)$, we add $e \in E'$ with capacity $c(e) - f(e)$.
3. **Backward Edges**: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we add $(v, u) \in E'$ with capacity $f(e)$.

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Residual Graph Example

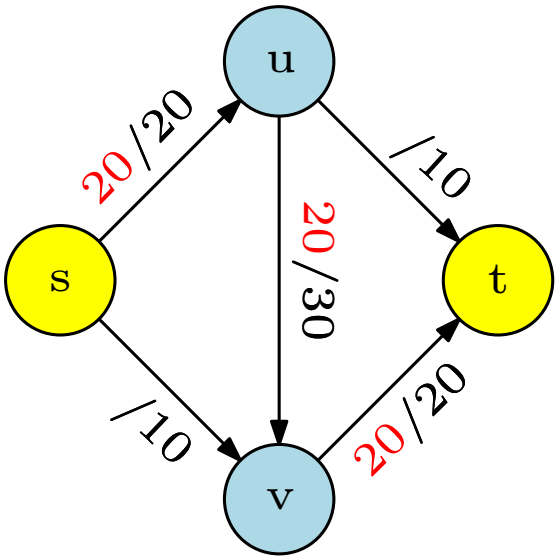

Figure: Flow on edges is indicated in red

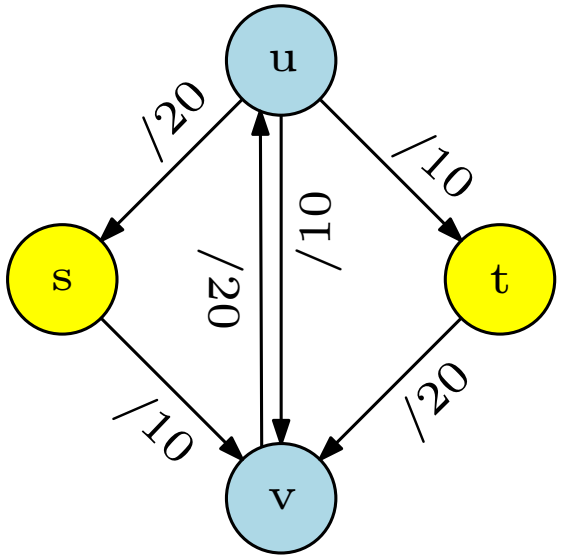

Figure: Residual Graph

Residual capacity

1. f flow f in network G.
2. c capacities on the edges.
3. The **residual capacity** is the leftover capacity on each edge. Formally:

$$c_f\Big((u,v)\Big) = \begin{cases} c(u,v) - f(u,v) & (u,v) \in E(G) \\ -f(v,u) & (v,u) \in E(G) \end{cases}$$

4. ...assumed that G does not contain both (u,v) and (v,u).
5. G_f with c_f is a new instance of network flow!

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Residual graph properties

1. **Observation:** Residual graph captures the "residual" problem exactly.
2. Flow in residual graph improves overall flow:

Lemma

Let f be a flow in G and G_f be the residual graph. If f' is a flow in G_f then $f + f'$ is a flow in G of value $v(f) + v(f')$.

3. If there is a bigger flow, we will find it:

Lemma

Let f and f' be two flows in G with $v(f') \geq v(f)$. Then there is a flow f'' of value $v(f') - v(f)$ in G_f.

4. Definition of $+$ and $-$ for flows is intuitive and the above lemmas are easy in some sense but a bit messy to formally prove.

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Residual Graph Property: Implication

Recursive algorithm for finding a maximum flow:

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MaxFlow(G, s, t):
    if the flow from s to t is 0 then
        return 0
    Find any flow f with v(f) > 0 in G
    Recursively compute a maximum flow f' in G_f
    Output the flow f + f'
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Iterative algorithm for finding a maximum flow:

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MaxFlow(G, s, t):
    Start with flow f that is 0 on all edges
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Residual capacity of an augmenting path

1. f current flow in G_f.
2. π: A path π in residual graph G_f.
3. c_f: Residual capacities in G_f.
4. The **residual capacity** of π is

$$c_f(\pi) = \min_{e \in E(\pi)} c_f(e).$$

5. $c_f(\pi)$ = maximum amount of flow that can be pushed on π in G_f without violating capacities (i.e., c_f).

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Ford-Fulkerson Algorithm

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    G_f is residual graph of G with respect to f
    while G_f has a simple s-t path do
        let P be simple s-t path in G_f
        f = augment(f, P)
        Construct new residual graph G_f.
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augment(f, P)
    let b be bottleneck capacity,
        i.e., min capacity of edges in P (in G_f)
    for each edge (u, v) in P do
        if e = (u, v) is a forward edge then
            f(e) = f(e) + b
        else (* (u, v) is a backward edge *)
            let e = (v, u) (* (v, u) is in G *)
            f(e) = f(e) − b
    return f
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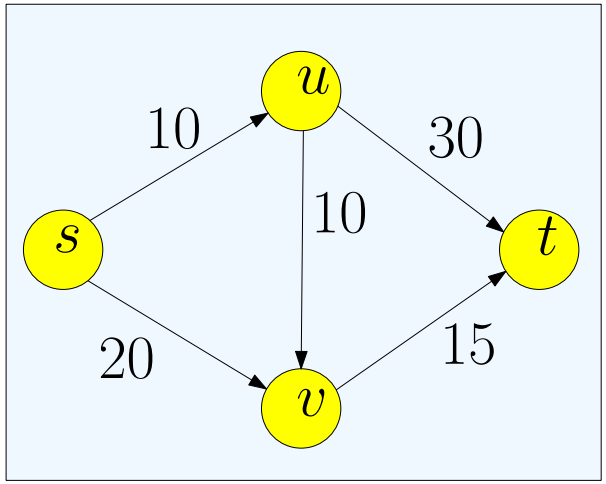

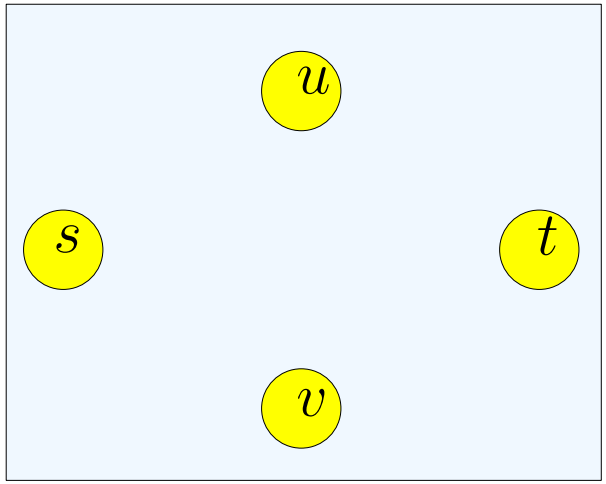

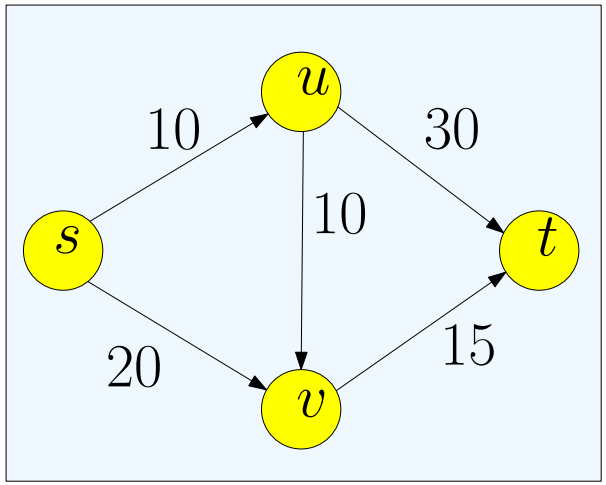

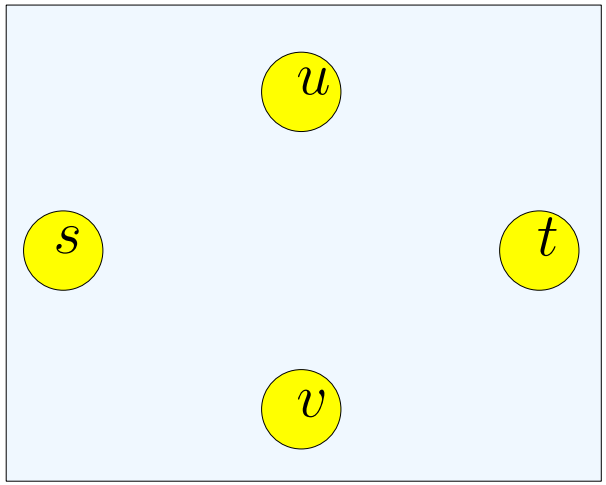

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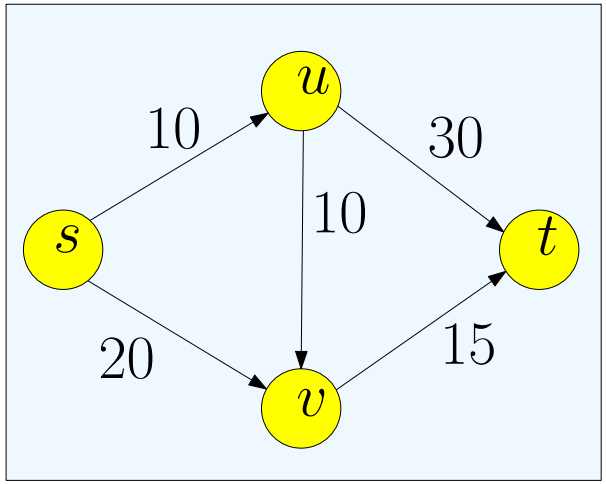

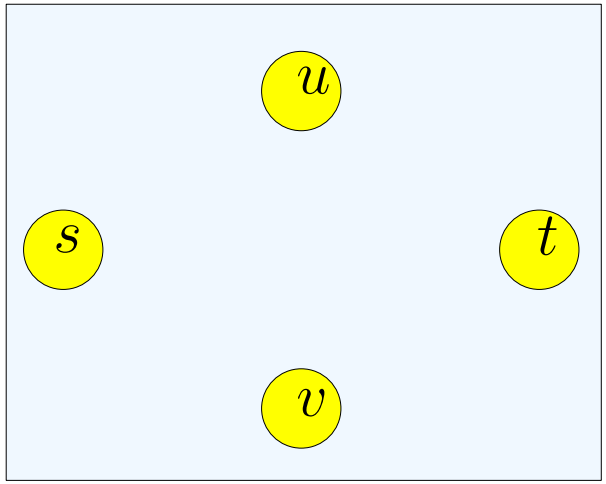

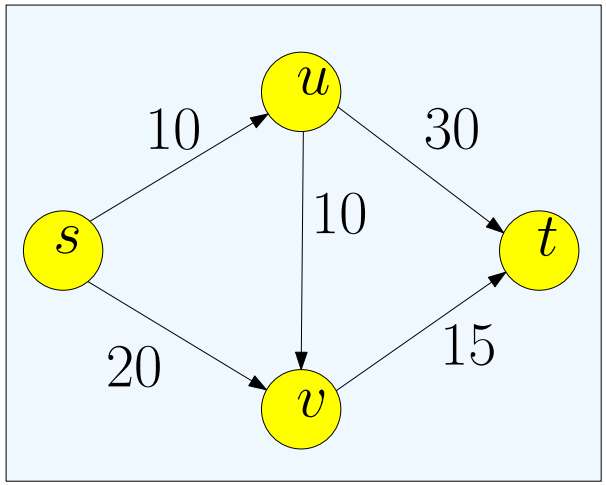

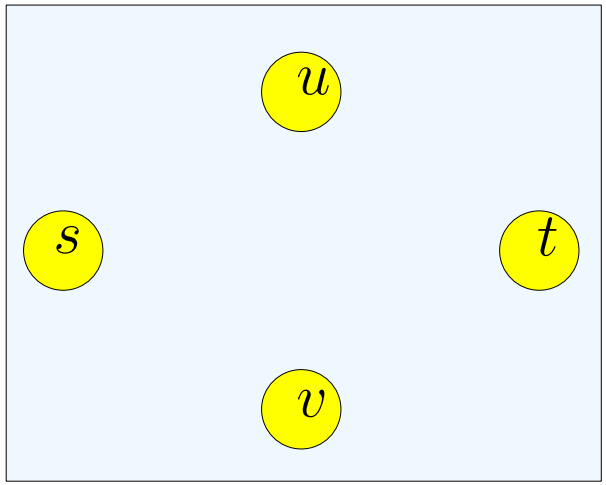

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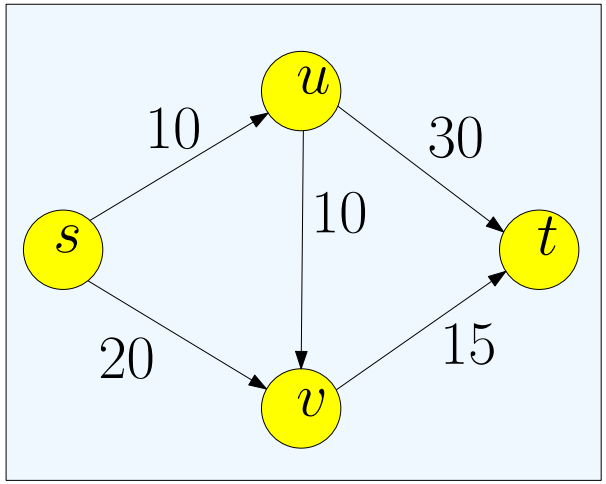
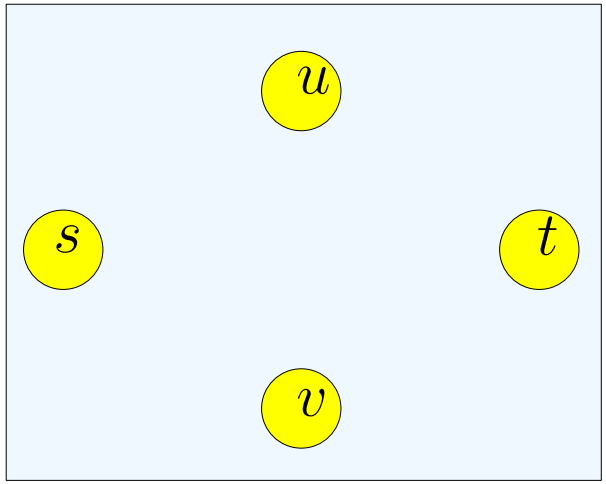
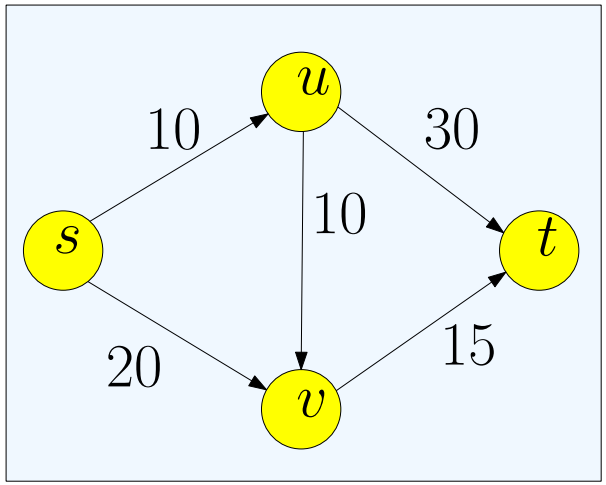
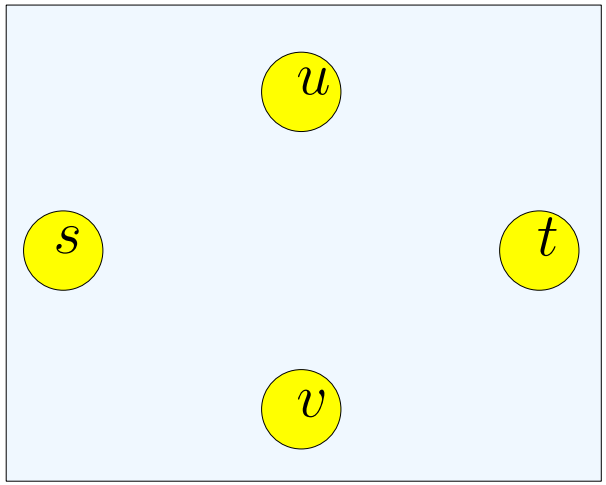
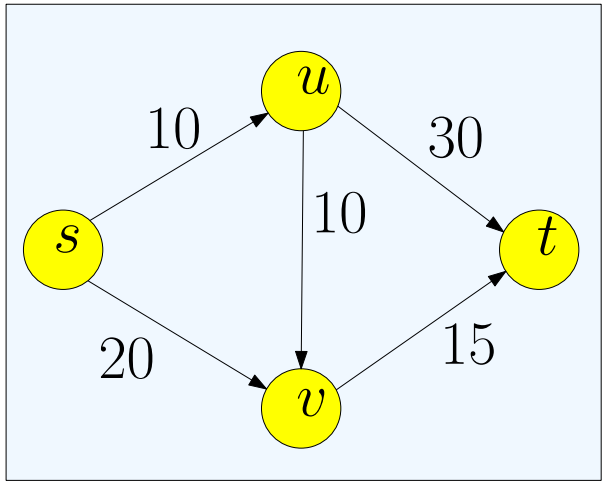


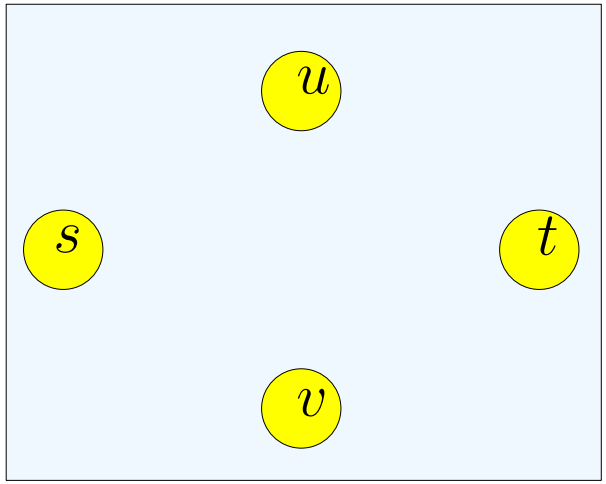


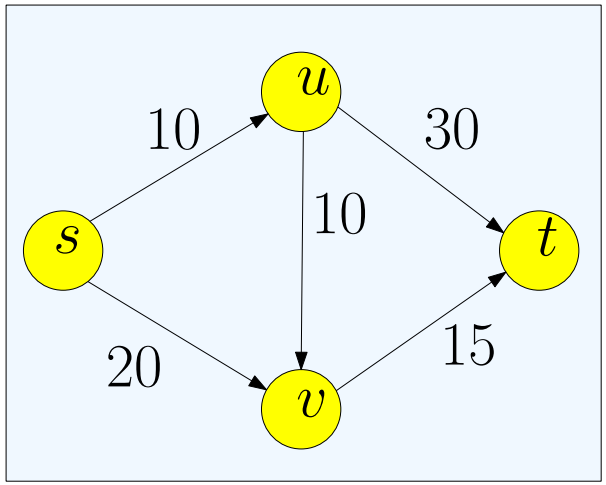


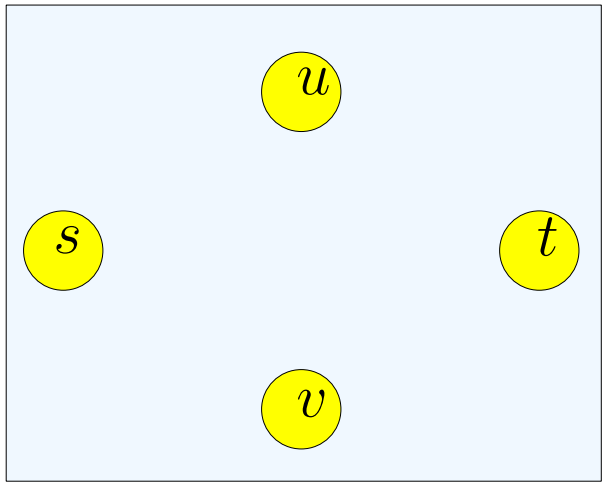


Example continued









Properties about Augmentation: Flow

Lemma

If f is a flow and P is a simple s-t path in G_f, then $f' = \text{augment}(f, P)$ is also a flow.

Proof.

Verify that f' is a flow. Let b be augmentation amount.

1. Capacity constraint: If $(u, v) \in P$ is a forward edge then $f'(e) = f(e) + b$ and $b \leq c(e) - f(e)$. If $(u, v) \in P$ is a backward edge, then letting $e = (v, u)$, $f'(e) = f(e) - b$ and $b \leq f(e)$. Both cases $0 \leq f'(e) \leq c(e)$.
2. Conservation constraint: Let v be an internal node. Let e_1, e_2 be edges of P incident to v. Four cases based on whether e_1, e_2 are forward or backward edges. Check cases (see fig next slide).

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Properties of Augmentation

Conservation Constraint

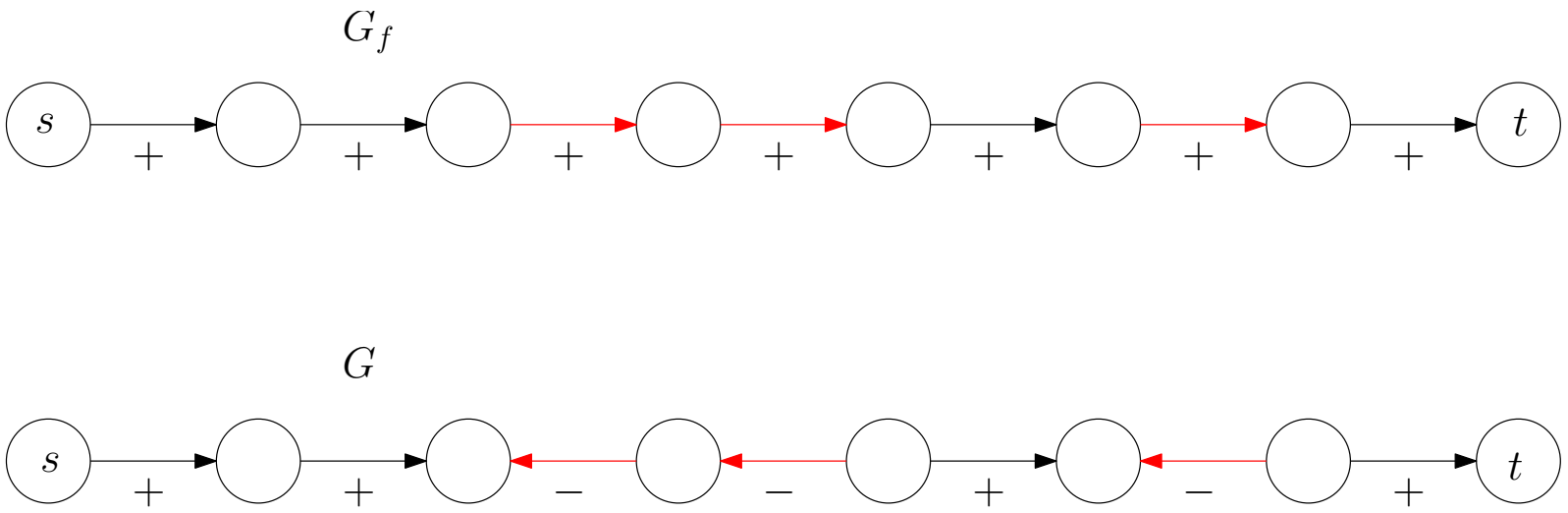

Figure: Augmenting path P in G_f and corresponding change of flow in G. Red edges are backward edges.

Properties of Augmentation

Integer Flow

Lemma

At every stage of the Ford-Fulkerson algorithm, the flow values on the edges (i.e., $f(e)$, for all edges e) and the residual capacities in G_f are integers.

Proof.

Initial flow and residual capacities are integers. Suppose lemma holds for j iterations. Then in $(j+1)$st iteration, minimum capacity edge b is an integer, and so flow after augmentation is an integer. □

Progress in Ford-Fulkerson

Proposition

Let f be a flow and f' be flow after one augmentation. Then $v(f) < v(f')$.

Proof.

Let P be an augmenting path, i.e., P is a simple s-t path in residual graph. We have the following.

1. First edge e in P must leave s.
2. Original network G has no incoming edges to s; hence e is a forward edge.
3. P is simple and so never returns to s.
4. Thus, value of flow increases by the flow on edge e. □

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Termination proof for integral flow

Theorem

Let C be the minimum cut value; in particular $C \leq \sum_{e \text{ out of } s} c(e)$. Ford-Fulkerson algorithm terminates after finding at most C augmenting paths.

Proof.

The value of the flow increases by at least 1 after each augmentation. Maximum value of flow is at most C. □

Running time

1. Number of iterations $\leq C$.
2. Number of edges in $G_f \leq 2m$.
3. Time to find augmenting path is $O(n + m)$.
4. Running time is $O(C(n + m))$ (or $O(mC)$).

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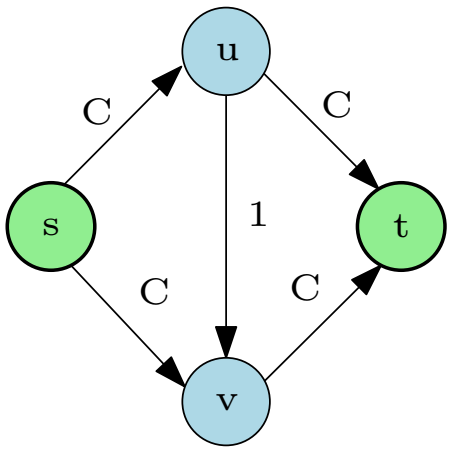
Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the running time be as $\Omega(mC)$ or is our analysis weak?

Efficiency of Ford-Fulkerson

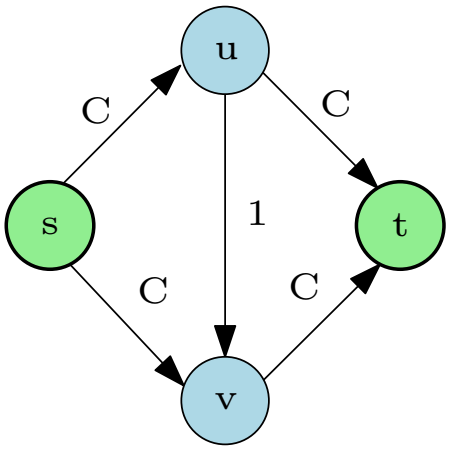
Flip-flop 1



Input network
+ capacities.

Efficiency of Ford-Fulkerson

Flip-flop 1



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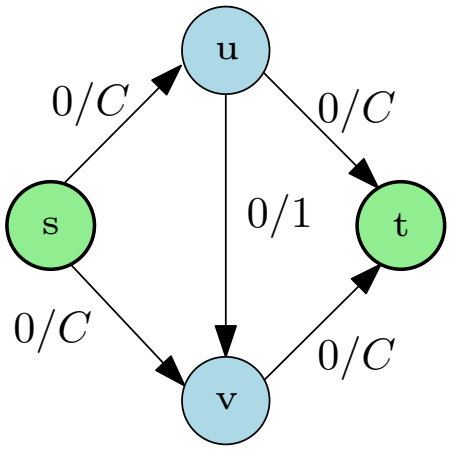
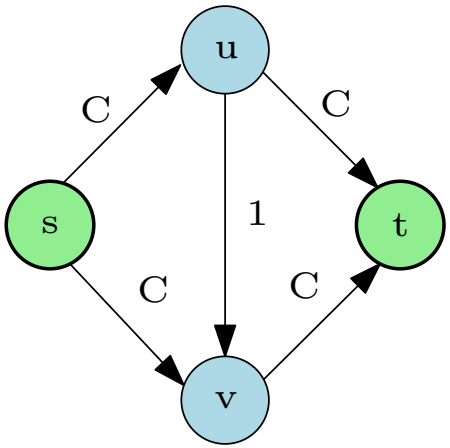
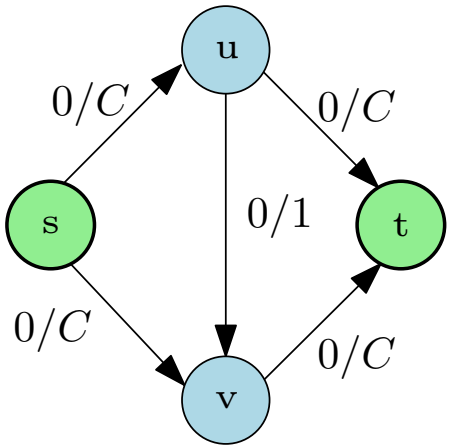
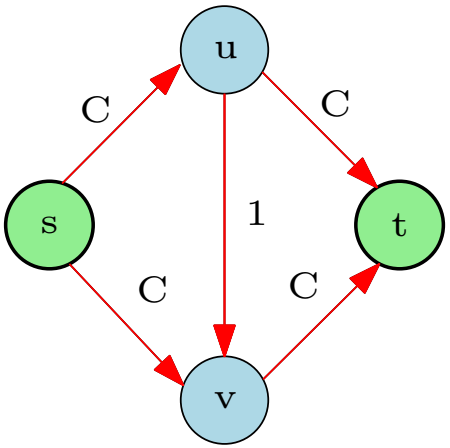


Initial empty flow

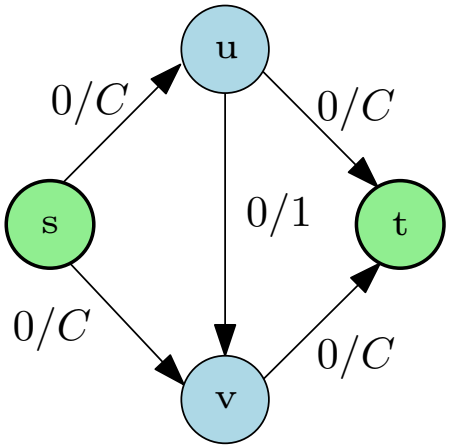
Efficiency of Ford-Fulkerson

Flip-flop 1

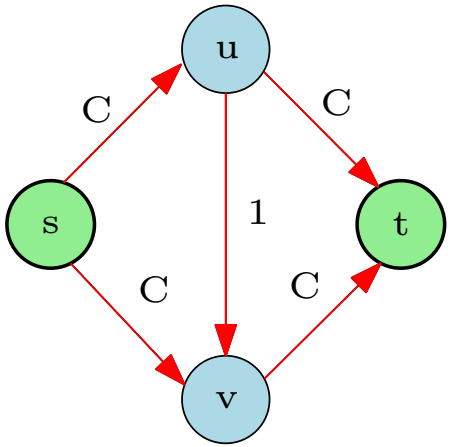
		
Input network + capacities.	Initial empty flow	Residual graph (Boring)

Efficiency of Ford-Fulkerson

Flip-flop 2



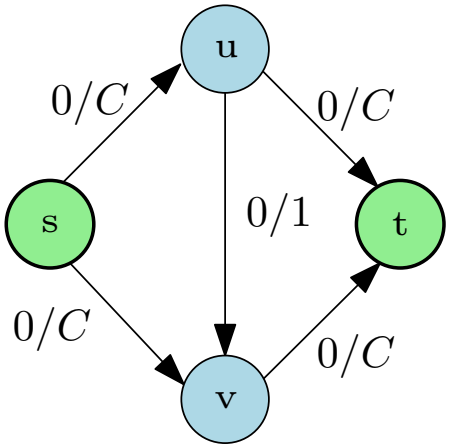
f_0: Initial empty flow



Residual graph

Efficiency of Ford-Fulkerson

Flip-flop 2



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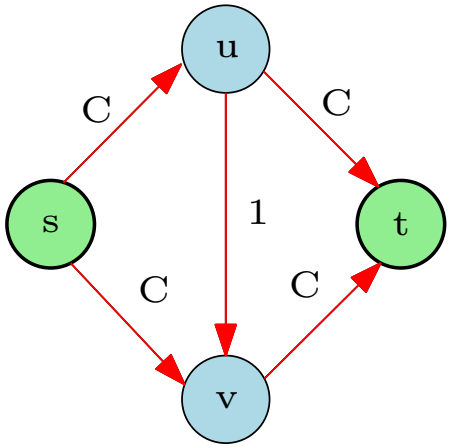

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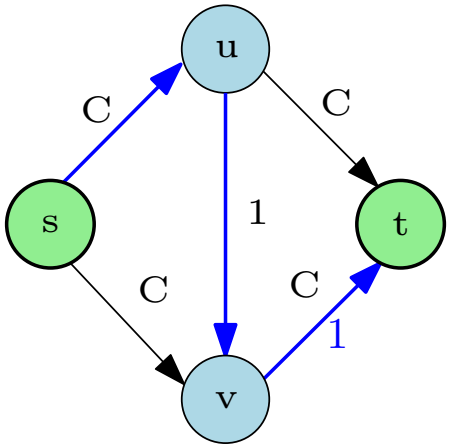
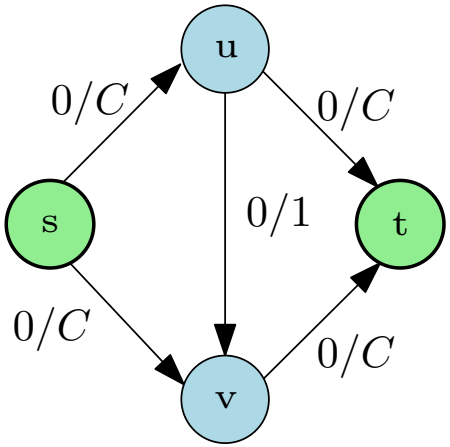


Augmenting path

Efficiency of Ford-Fulkerson

Flip-flop 2



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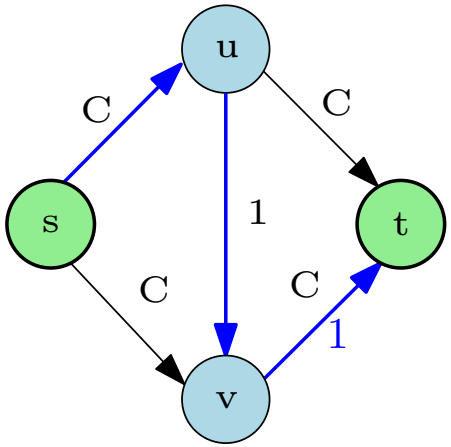

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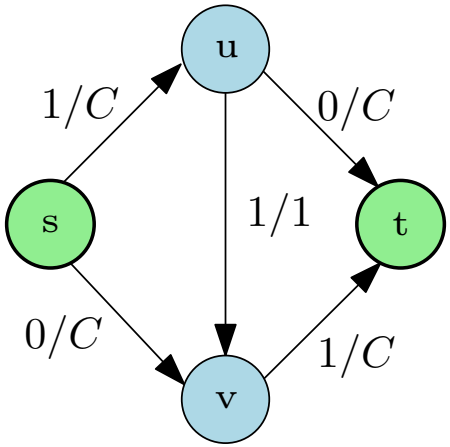
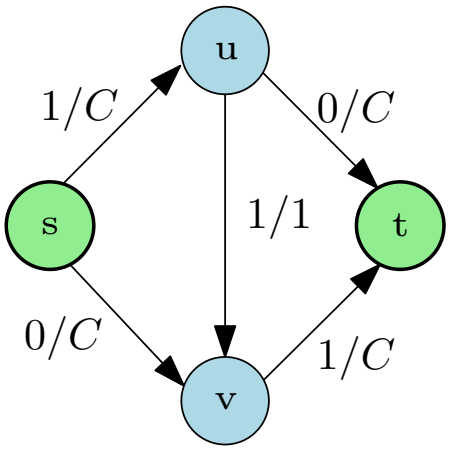


f_1: Updated flow

Efficiency of Ford-Fulkerson

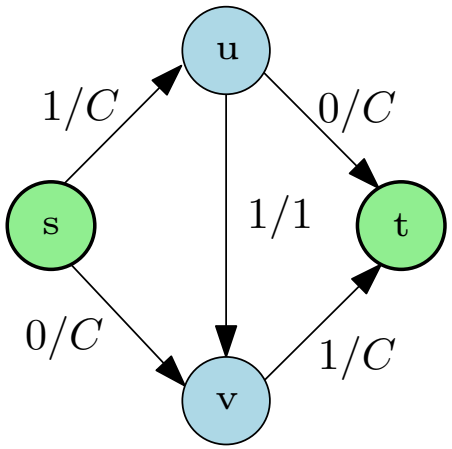
Flip-flop 3



f_1: current flow

Efficiency of Ford-Fulkerson

Flip-flop 3



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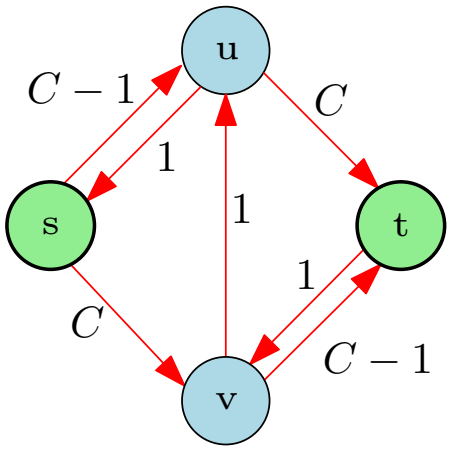
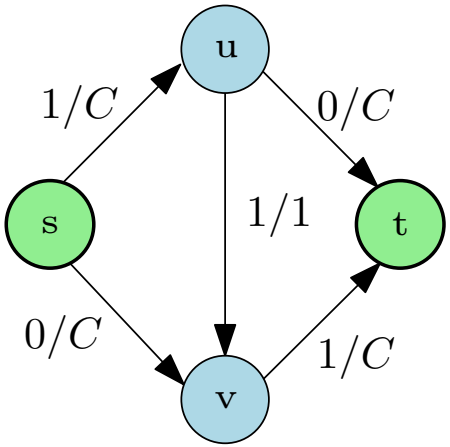


G_{f_1}: Residual graph

Efficiency of Ford-Fulkerson

Flip-flop 3



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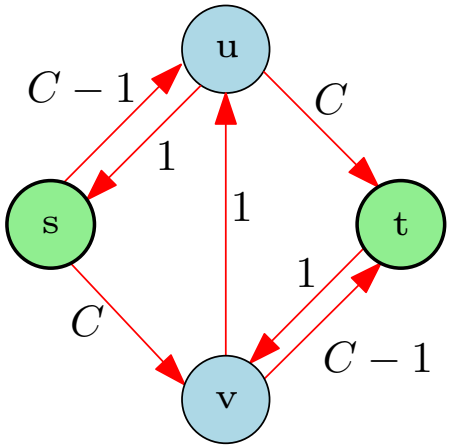

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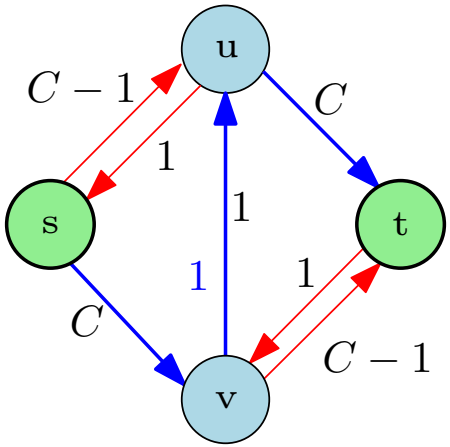
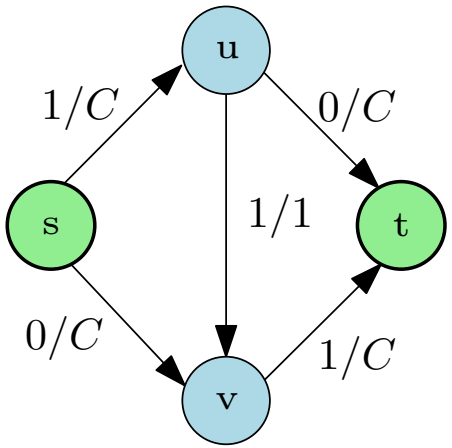


Augmenting path

Efficiency of Ford-Fulkerson

Flip-flop 3



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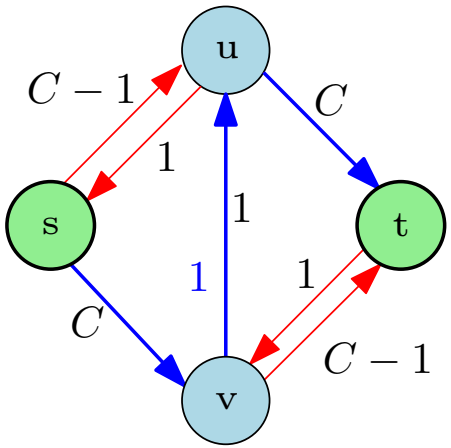

Augmenting path

Efficiency of Ford-Fulkerson

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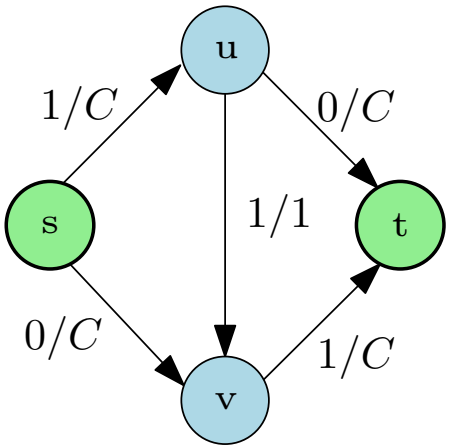

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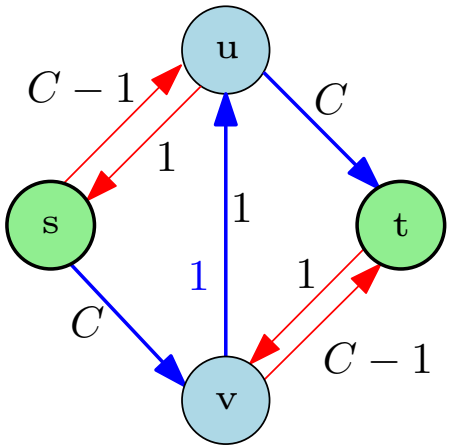
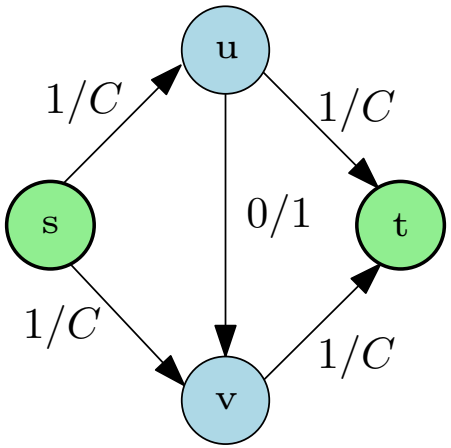


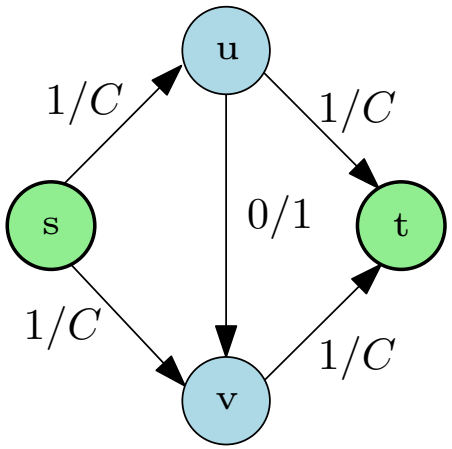
Augmenting path



f_2: New flow

Efficiency of Ford-Fulkerson

Flip-flop 4



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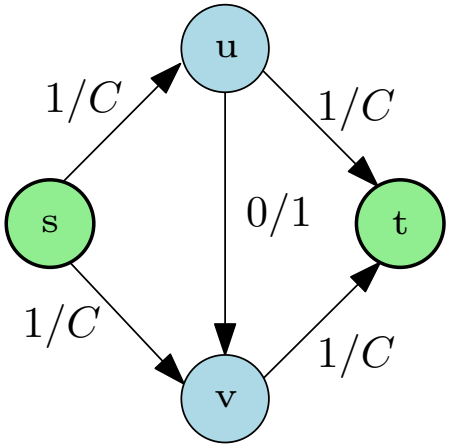
Efficiency of Ford-Fulkerson

Flip-flop 4



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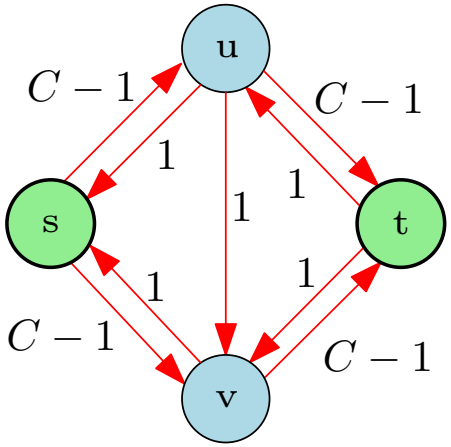
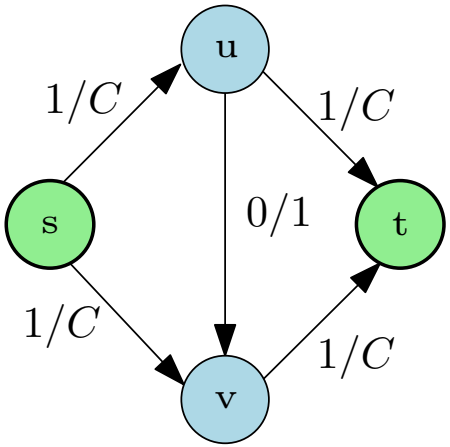


G_{f_2}: Residual graph

Efficiency of Ford-Fulkerson

Flip-flop 4



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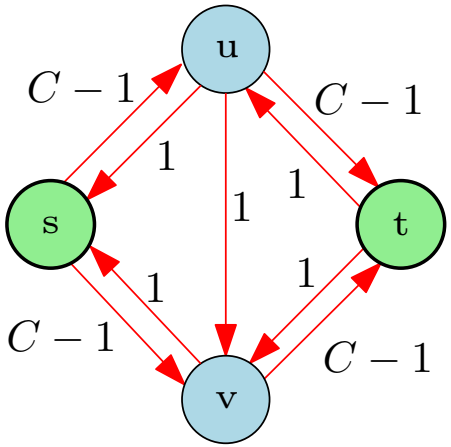

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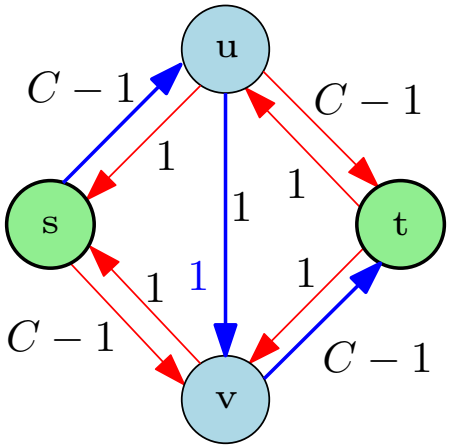
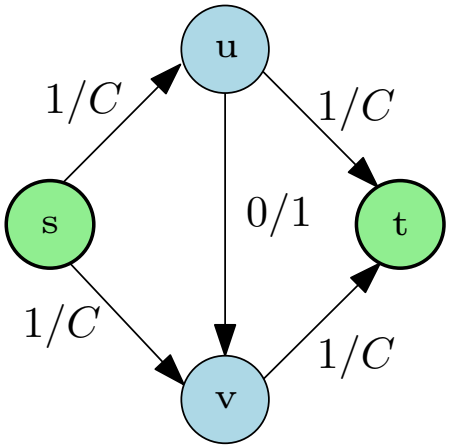


Augmenting path

Efficiency of Ford-Fulkerson

Flip-flop 4



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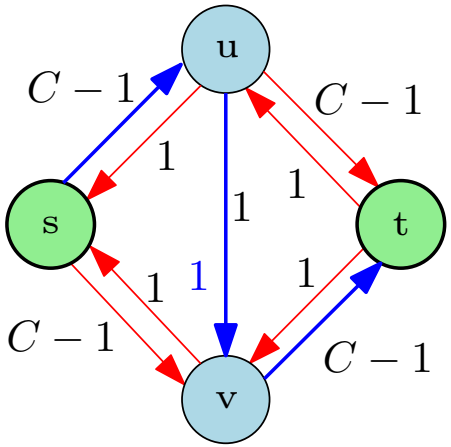

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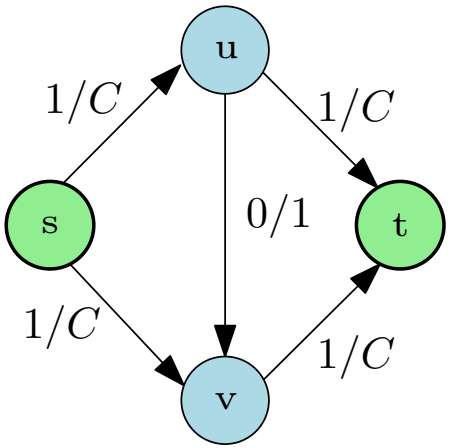

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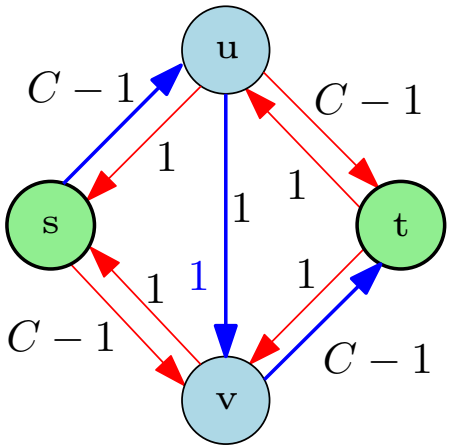

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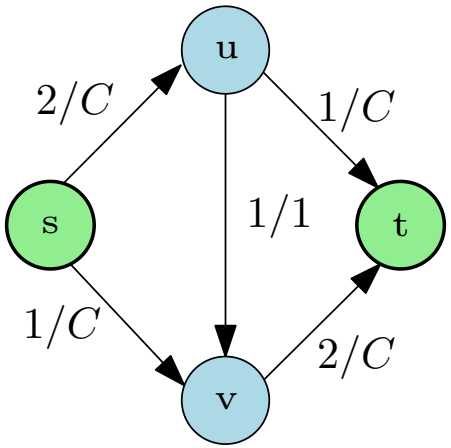
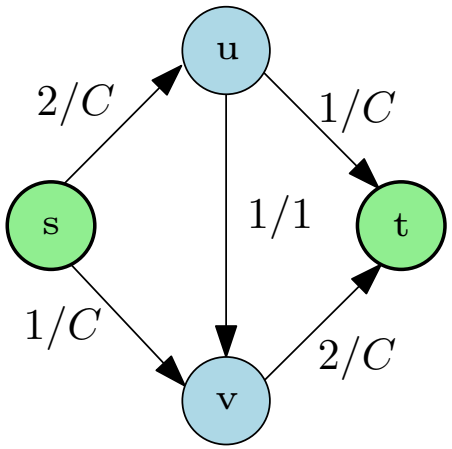


f_3: New flow

Efficiency of Ford-Fulkerson

Flip-flop 5



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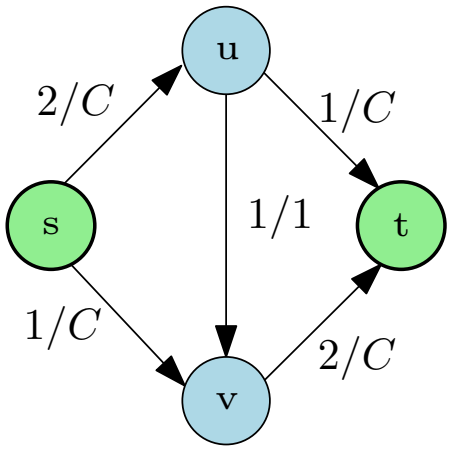
Efficiency of Ford-Fulkerson

Flip-flop 5



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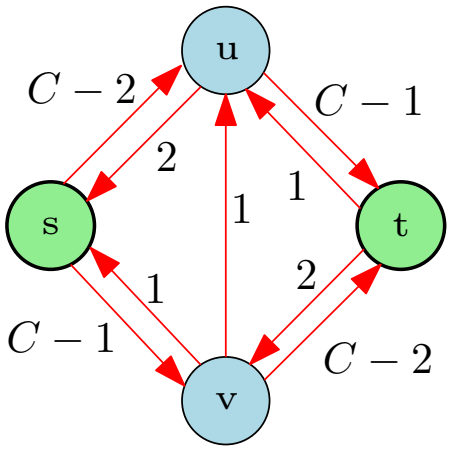
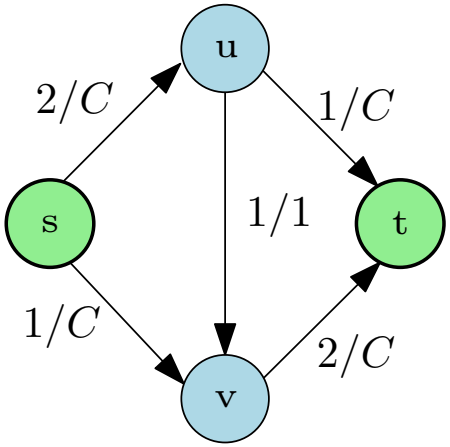


G_{f_3}: Residual graph

Efficiency of Ford-Fulkerson

Flip-flop 5



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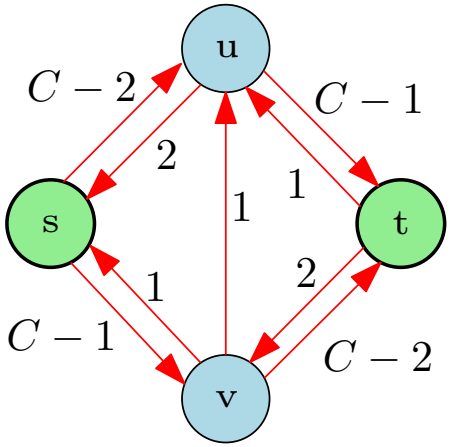

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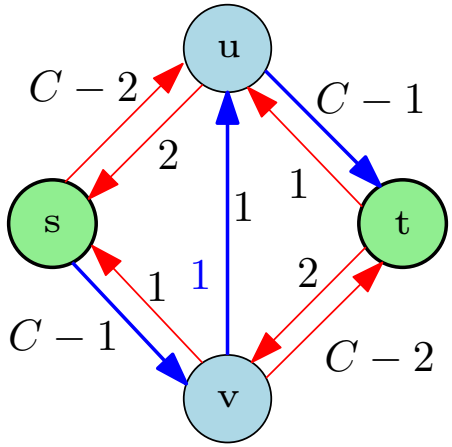
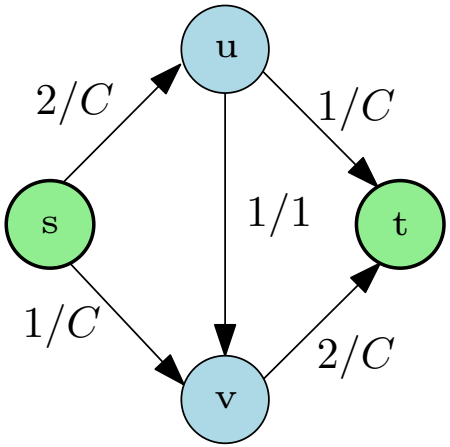


Augmenting path

Efficiency of Ford-Fulkerson

Flip-flop 5



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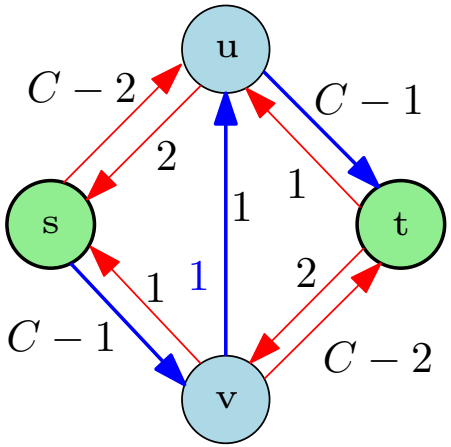

Augmenting path

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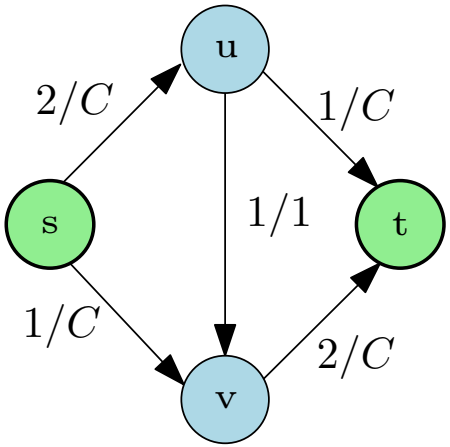
Flip-flop 5



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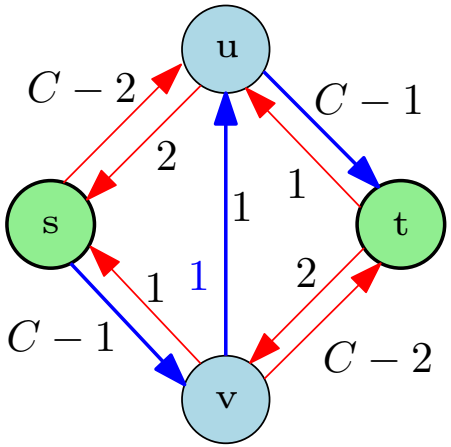

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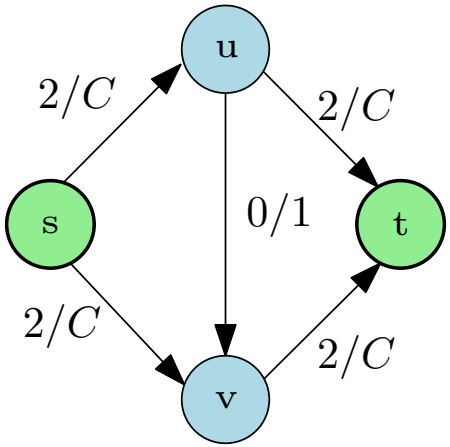
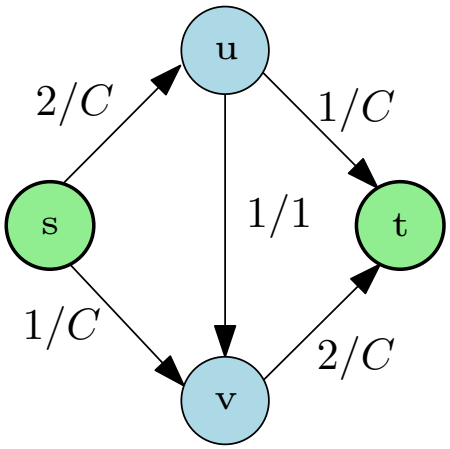


f_4: New flow

Efficiency of Ford-Fulkerson

Flip-flop 5



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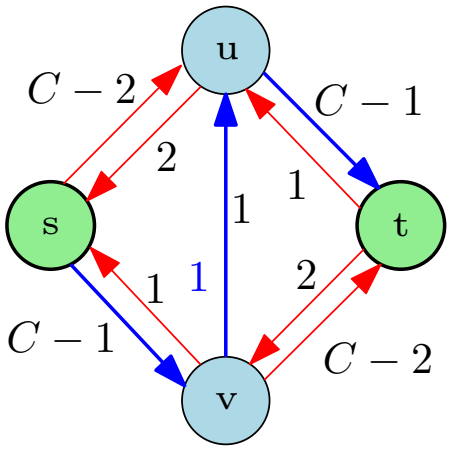

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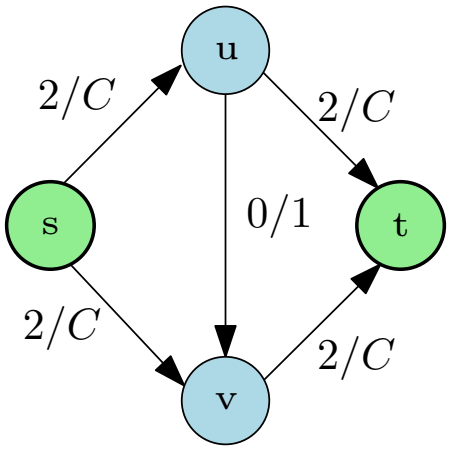

f_4: New flow

And so it continues for $2C$ iterations...

Efficiency of Ford-Fulkerson

1. Running time $= O(mC)$ is not polynomial.
2. Can the running time be as $\Omega(mC)$ or is our analysis weak?
3. Previous example shows this is tight!.
4. Ford-Fulkerson can take $\Omega(C)$ iterations.

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Correctness of Ford-Fulkerson

Why the augmenting path approach works

1. Question: When the algorithm terminates, is the flow computed the maximum s-t flow?
2. Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

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Recalling Cuts

1. Definition:

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Given a flow network an **s-t cut** is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \rightarrow t$ path in $E - E'$. **Capacity** of cut E' is $\sum_{e \in E'} c(e)$.

2. **Vertex cut**: Let $A \subset V$ such that
   1. $s \in A$, $t \notin A$, and
   2. $B = V \setminus -A$ and hence $t \in B$.
3. Define $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$

Claim

(A, B) *is an* s-t *cut.*

4. Recall: Every *minimal* s-t cut E' is a cut of the form (A, B).

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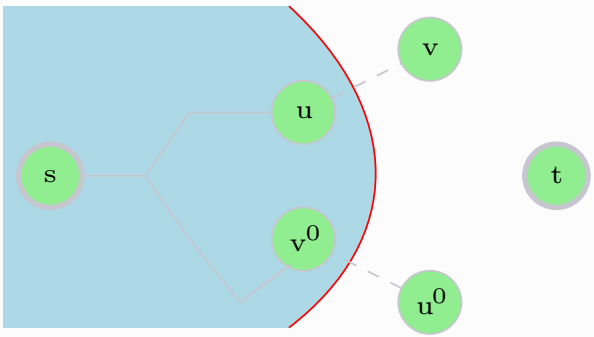
Ford-Fulkerson Correctness

Lemma

If there is no s-t path in G_f then there is some cut (A, B) such that $v(f) = c(A, B)$

Proof.

Let A be all vertices reachable from s in G_f; $B = V \setminus A$.



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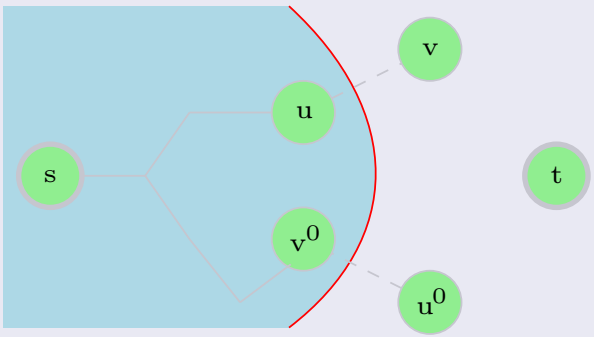

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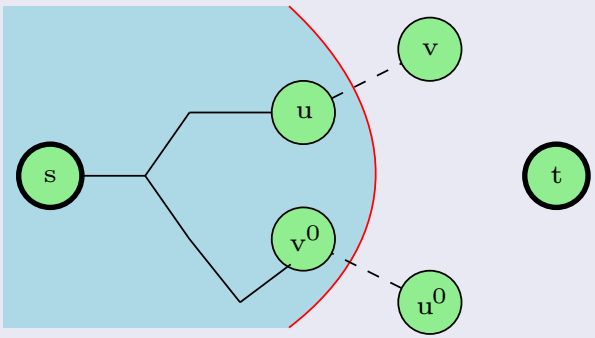

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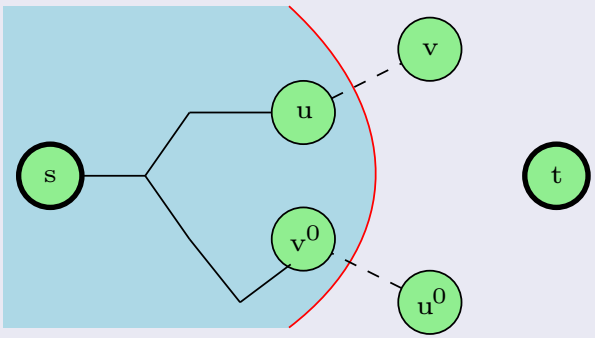

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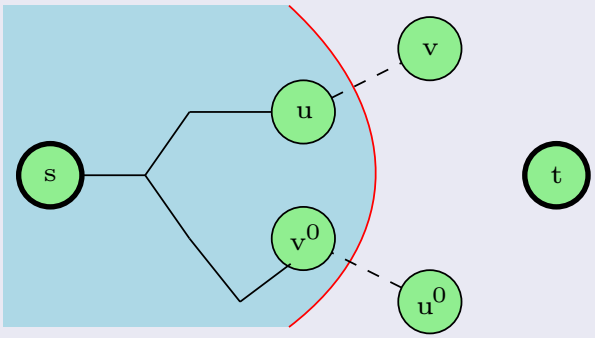
□

Lemma Proof Continued

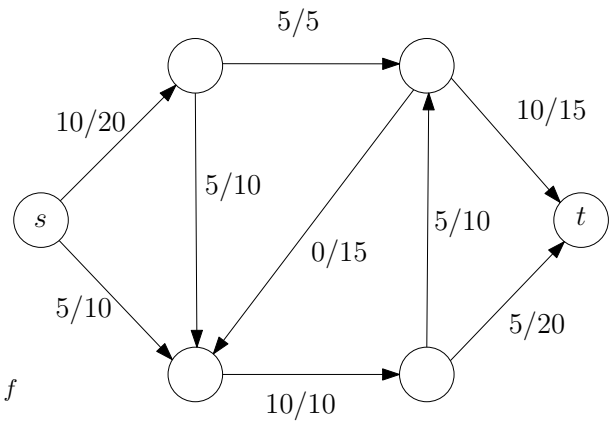
Proof.



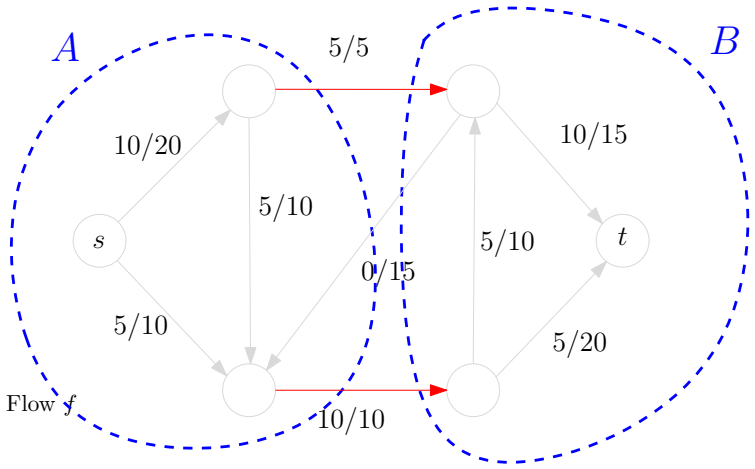
1. If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s in G_f
2. Thus,

$$\begin{aligned} v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\ &= f^{\text{out}}(A) - 0 \\ &= c(A, B) - 0 \\ &= c(A, B). \end{aligned}$$

Example



Flow f



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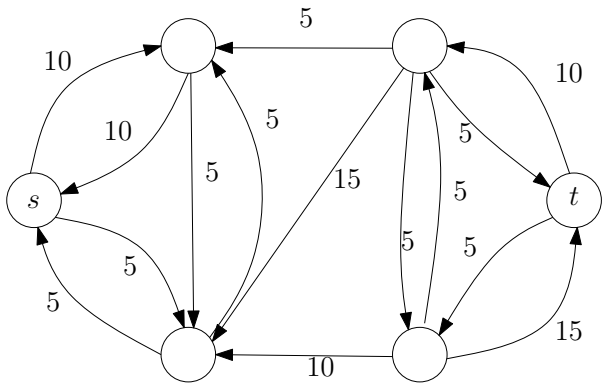
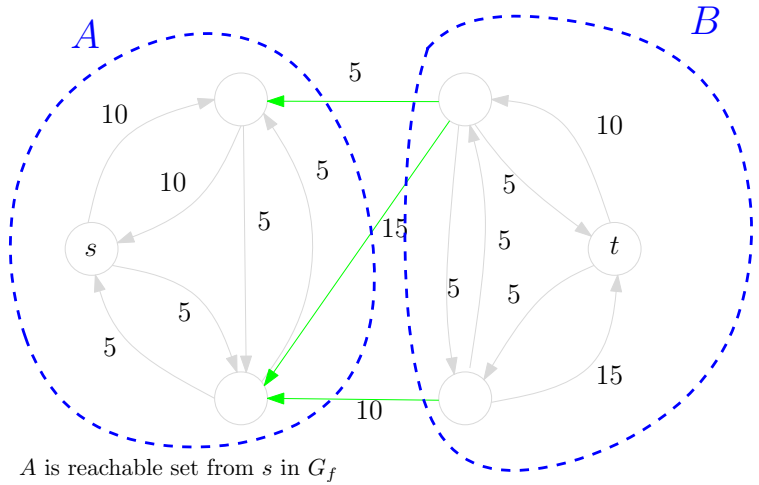
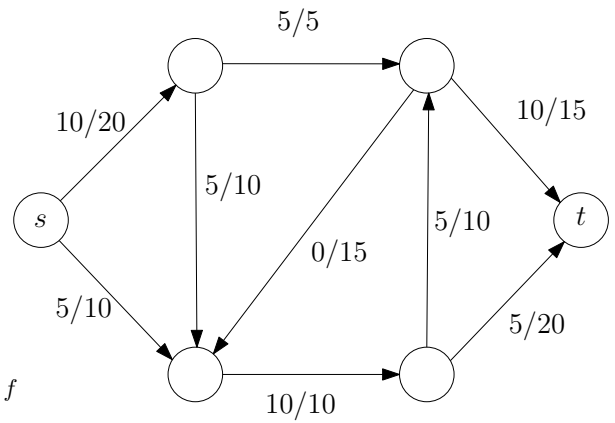


Residual graph G_f: no s-t path



A is reachable set from s in G_f

Example



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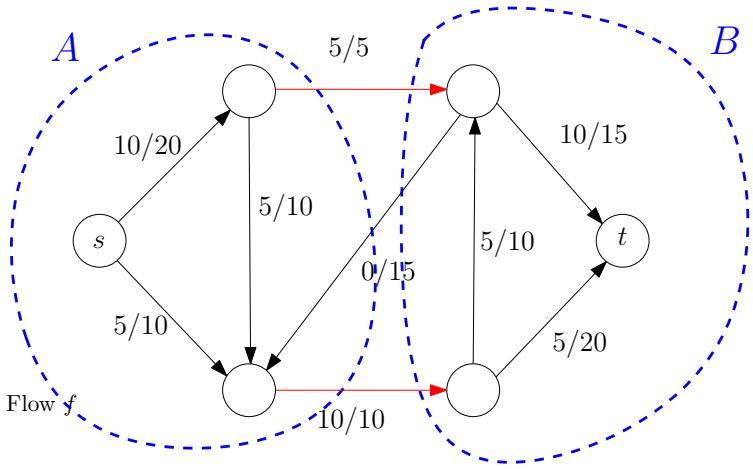
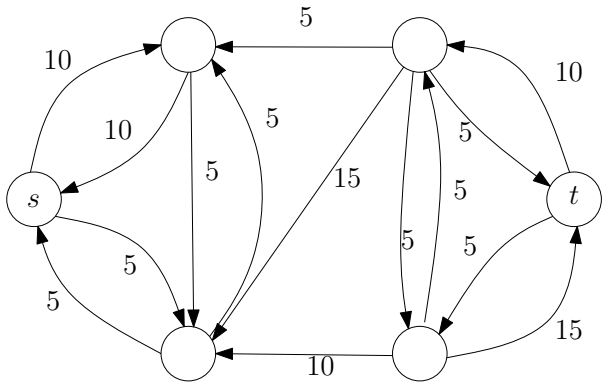
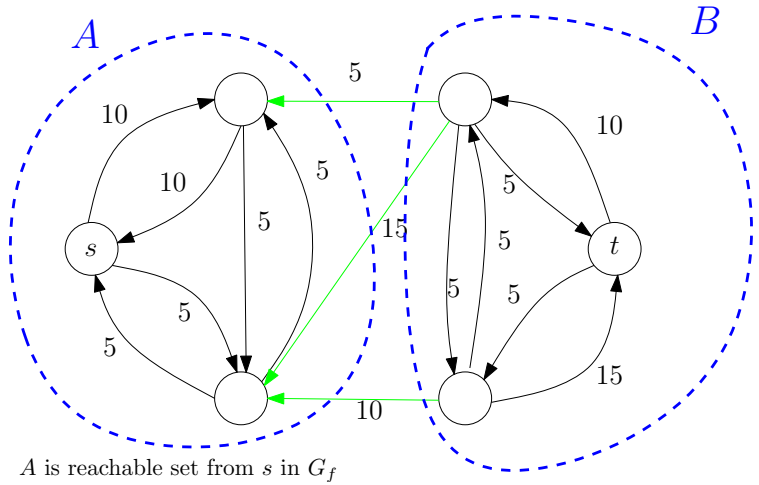


Flow f



Residual graph G_f: no s-t path



A is reachable set from s in G_f

Ford-Fulkerson Correctness

Theorem

The flow returned by the algorithm is the maximum flow.

Proof.

1. For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$.
2. For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-t cut (A^*, B^*).
3. Hence, f^* is maximum.

□

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G, the value of a maximum s-t flow is equal to the capacity of the minimum s-t cut.

Proof.

Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. □

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G with integer capacities, there is a maximum s-t flow that is integer valued.

Proof.

Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. □

Efficiency of Ford-Fulkerson

1. Running time $= O(mC)$ is not polynomial.
2. Can the upper bound be achieved?
3. Yes - saw an example.

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1. Question: Is there a polynomial time algorithm for max-flow?
2. Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way?
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Augmenting Paths with Large Bottleneck Capacity

1. Pick augmenting paths with largest bottleneck capacity in each iteration of Ford-Fulkerson.
2. How do we find path with largest bottleneck capacity?
   1. Assume we know Δ the bottleneck capacity
   2. Remove all edges with residual capacity $\leq \Delta$
   3. Check if there is a path from s to t
   4. Do binary search to find largest Δ
   5. Running time: $O(m \log C)$
3. Can we bound the number of augmentations? Can show that in $O(m \log C)$ augmentations the algorithm reaches a max flow. This leads to an $O(m^2 \log^2 C)$ time algorithm.

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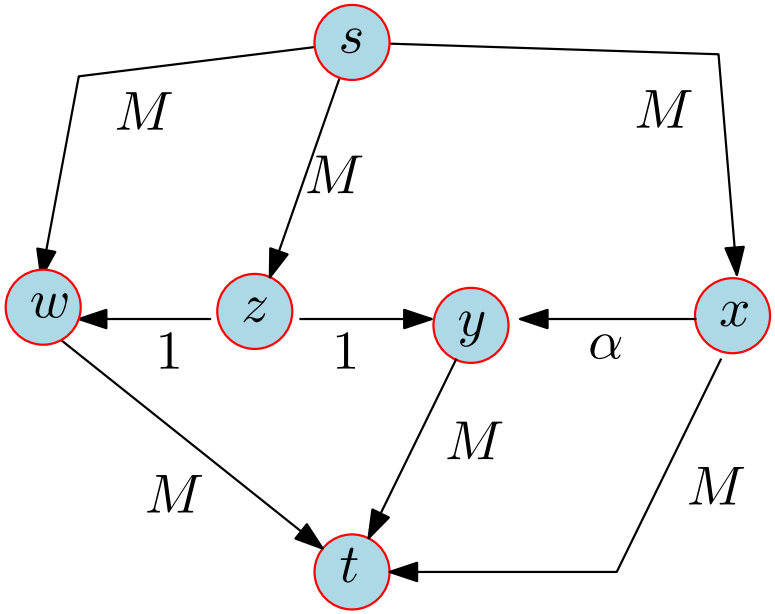
Removing Dependence on C

1. **Dinic [1970], Edmonds and Karp [1972]**
   Picking augmenting paths with fewest number of edges yields a $O(m^2 n)$ algorithm, i.e., independent of C. Such an algorithm is called a strongly polynomial time algorithm since the running time does not depend on the numbers (assuming RAM model). (Many implementation of Ford-Fulkerson would actually use shortest augmenting path if they use **BFS** to find an s-t path).
2. Further improvements can yield algorithms running in $O(mn \log n)$, or $O(n^3)$.

Part II

Not for lecture: Non-termination of Ford-Fulkerson

Ford-Fulkerson runs in vain



1. M: large positive integer.
2. $\alpha = (\sqrt{5} - 1)/2 \approx 0.618$.
3. $\alpha < 1$,
4. $1 - \alpha < \alpha$.
5. Maximum flow in this network is: $2M + 1$.

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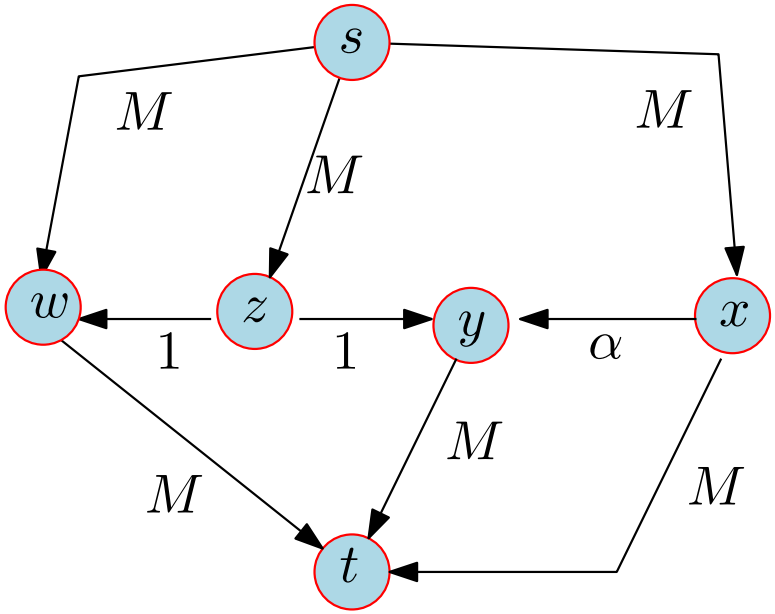

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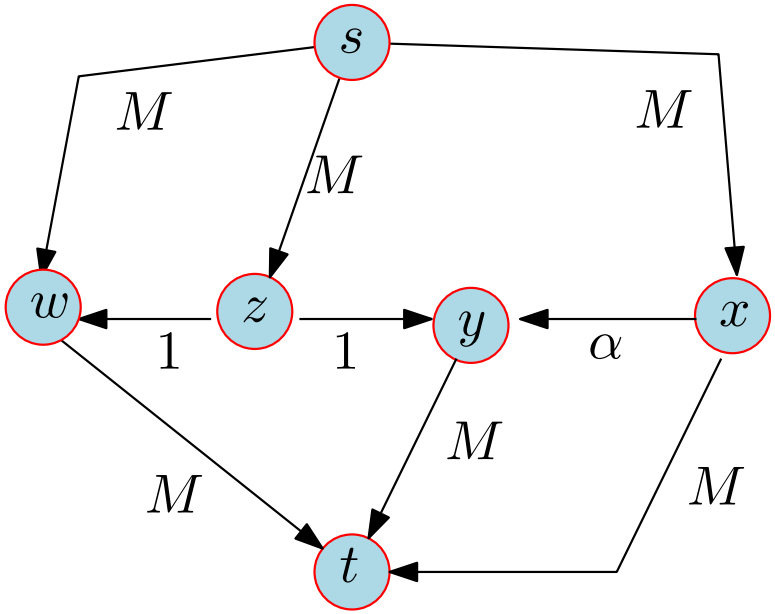

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For $\alpha = \frac{\sqrt{5}-1}{2}$:

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$$= 1-\frac{\sqrt{5}-1}{2}$$
$$= 1-\alpha.$$

Some algebra...

Claim

Given: $\alpha = (\sqrt{5} - 1)/2$ *and* $\alpha^2 = 1 - \alpha$.

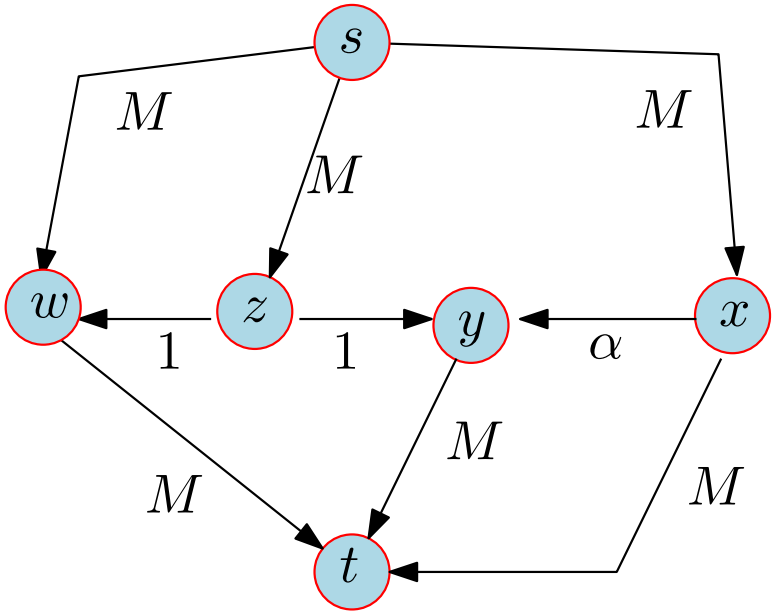
$$\Longrightarrow \forall i \qquad \alpha^i - \alpha^{i+1} = \alpha^{i+2}$$

Proof.

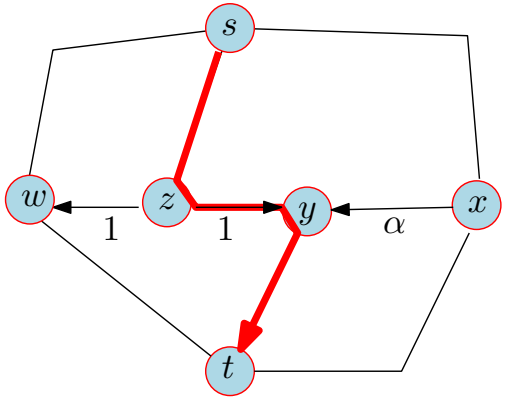
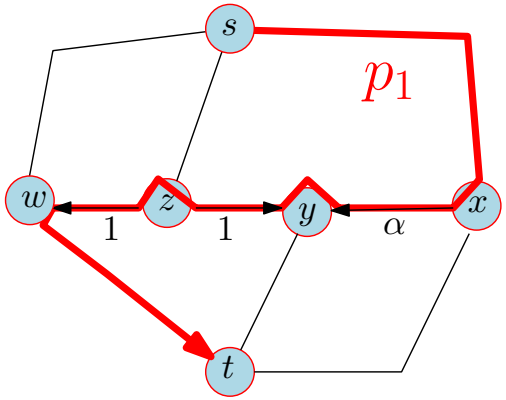
$$\alpha^i - \alpha^{i+1} = \alpha^i(1 - \alpha) = \alpha^i\alpha^2 = \alpha^{i+2}.$$

□

The network



Let it flow...

#	Augment. path π	c_π	New residual network
0.			
1.			

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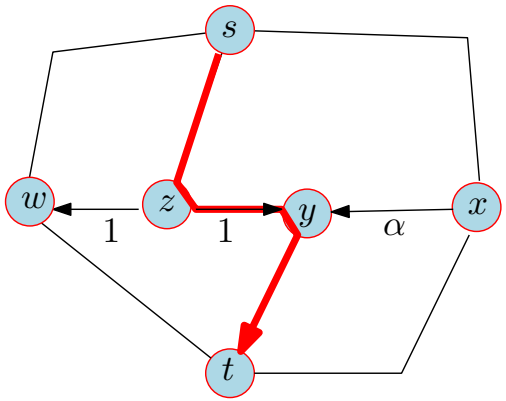
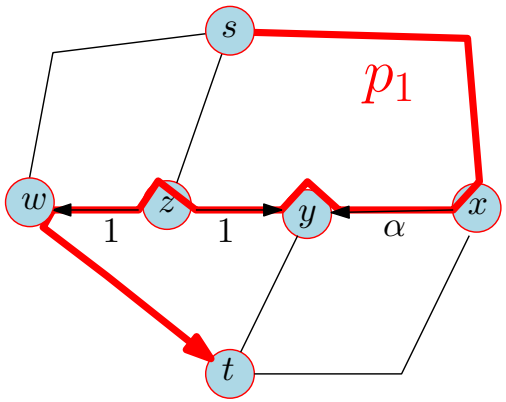
0.		**1**	
1.			

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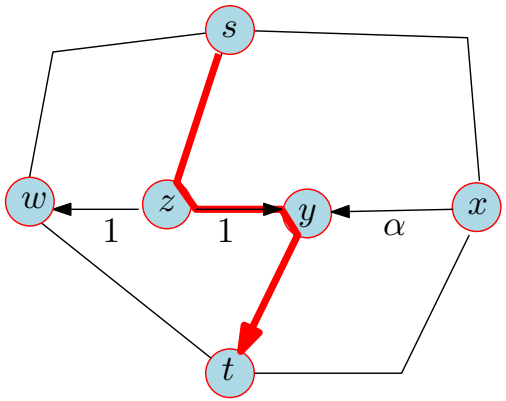
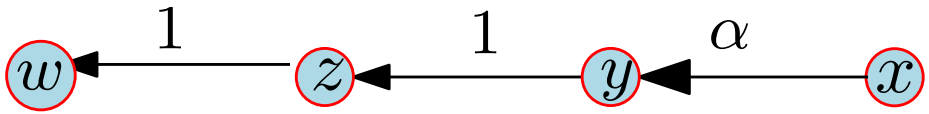
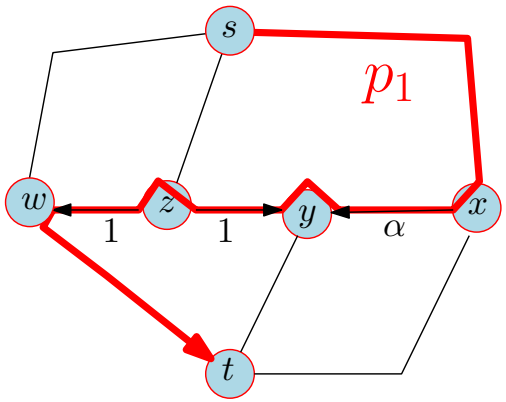
0.		**1**	
1.			

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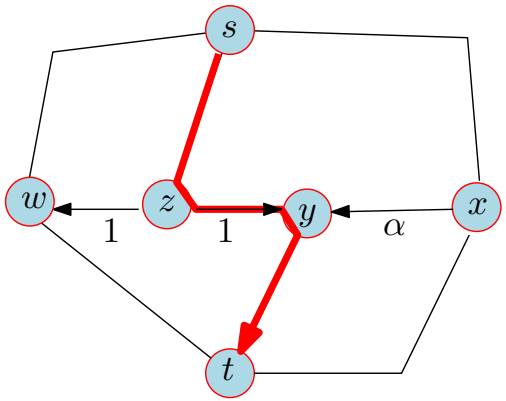
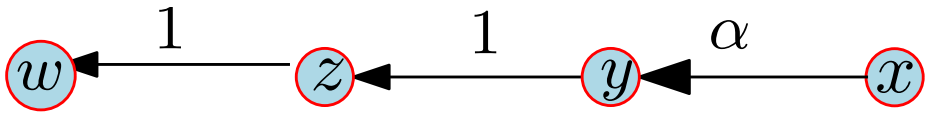
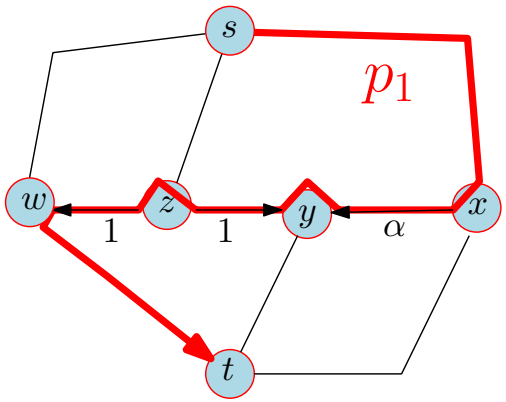
0.		$\mathbf{1}$	
1.		$\boldsymbol{\alpha}$	

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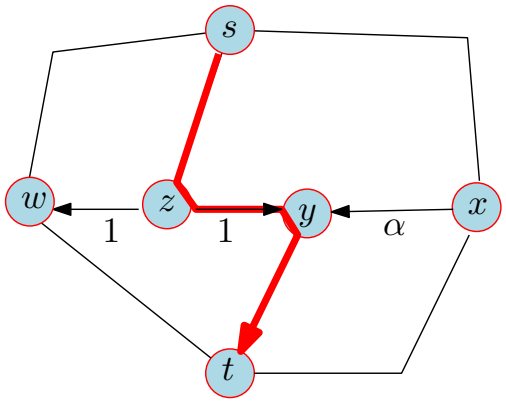
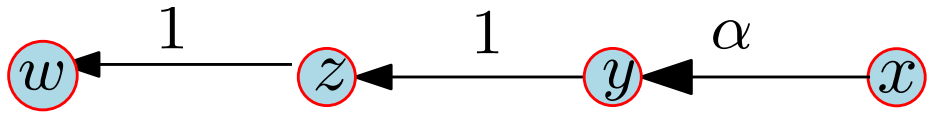
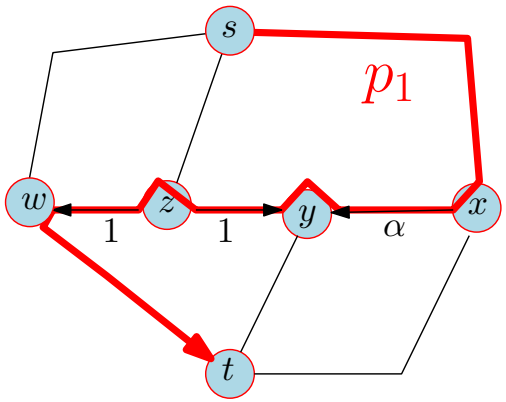
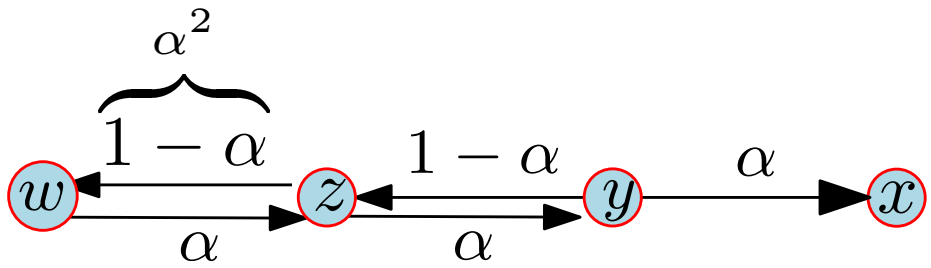
0.		$\mathbf{1}$	
1.		$\boldsymbol{\alpha}$	

Let it flow II

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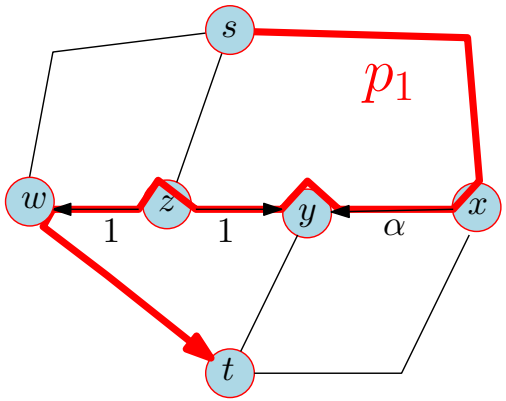
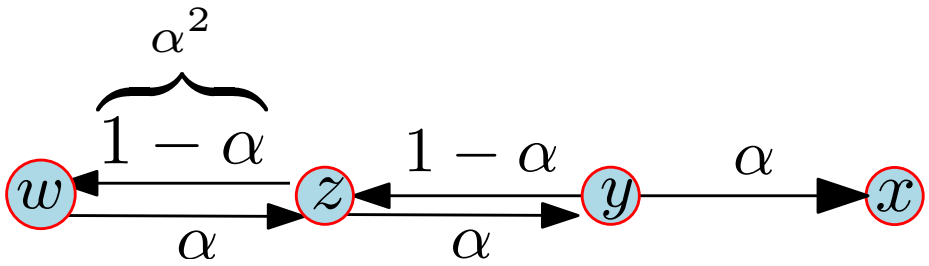
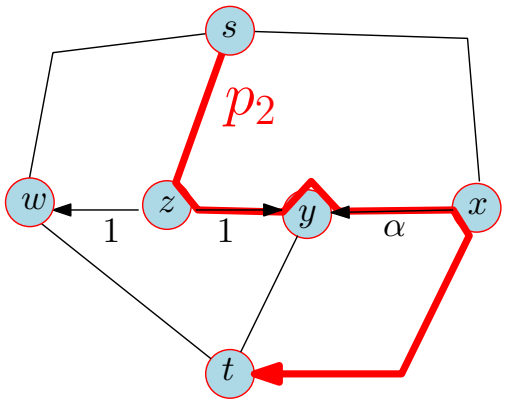
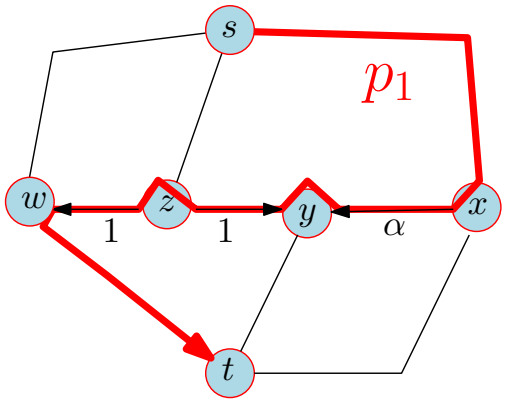
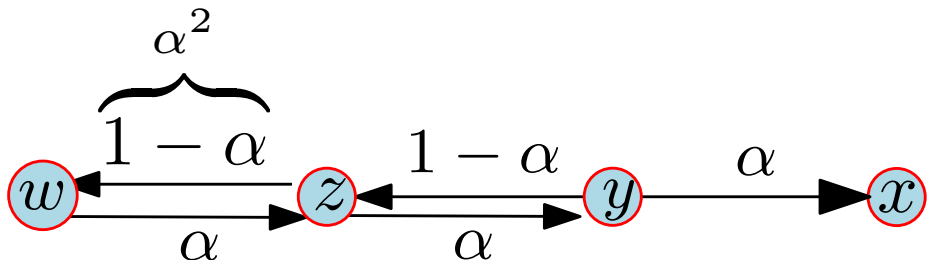
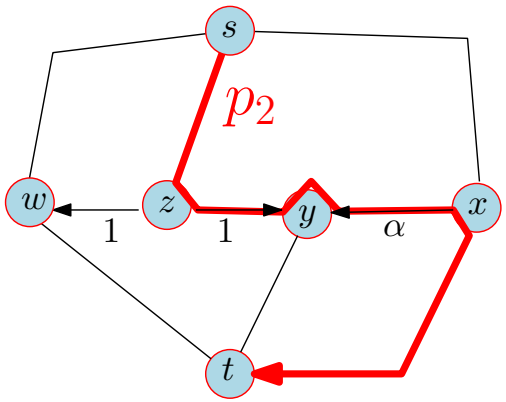
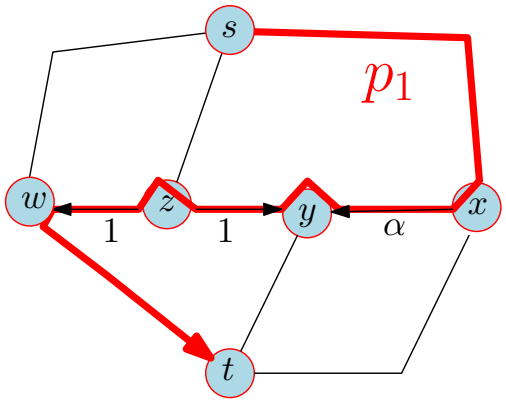
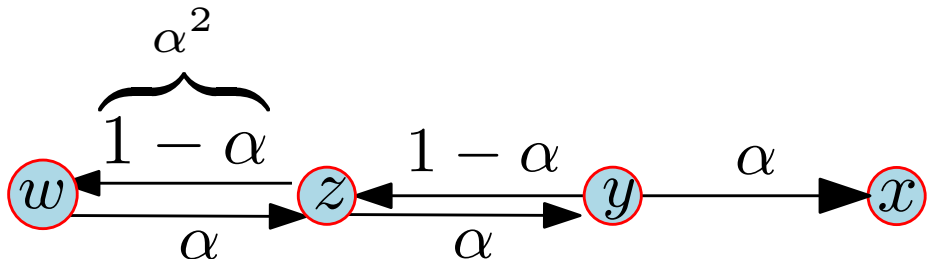
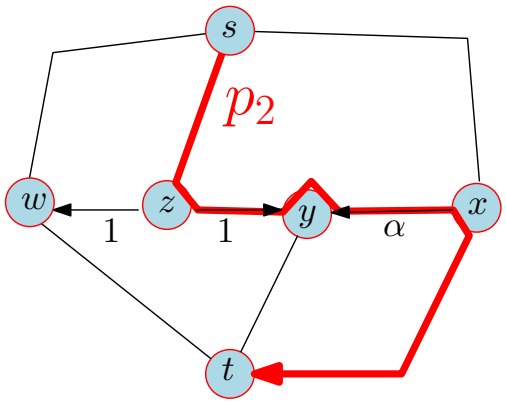
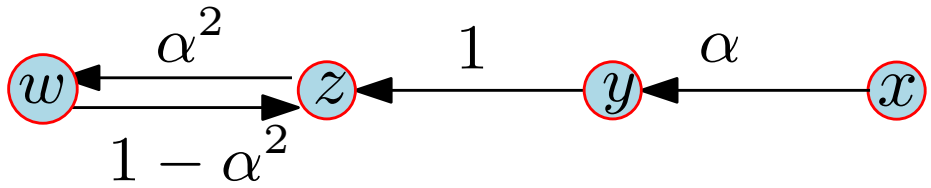
1.		α	
2.			

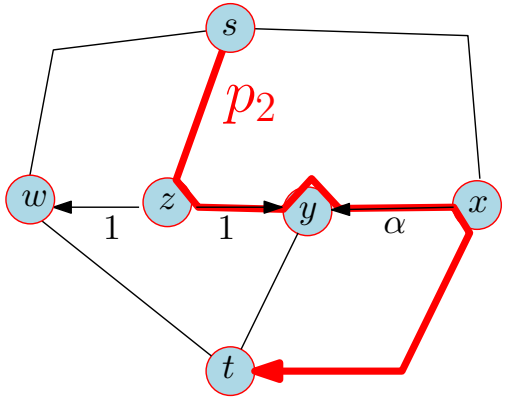
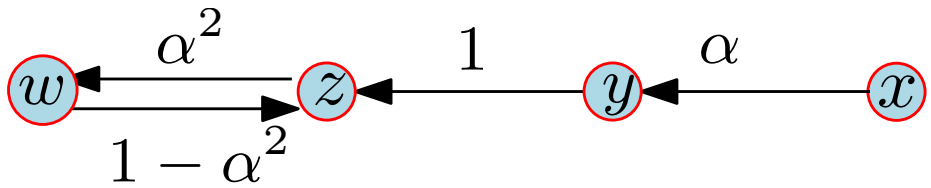
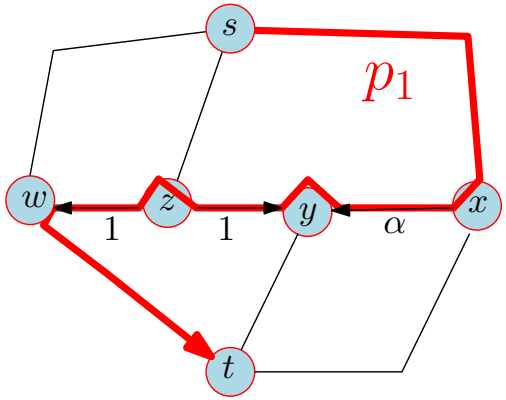
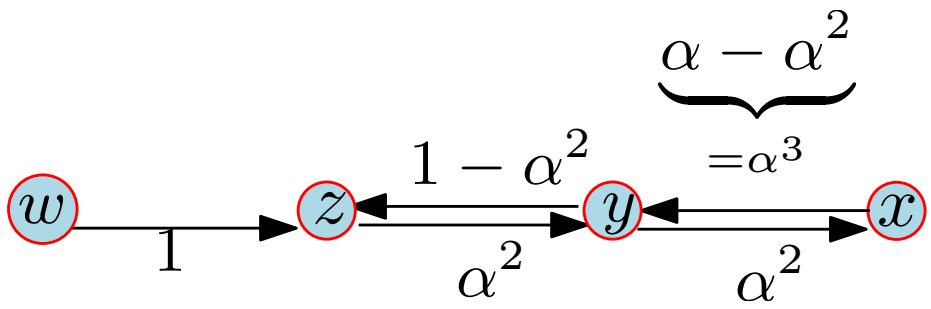
Let it flow II

#	Augment. path π	c_π	New residual network
1.		α	
2.		α	

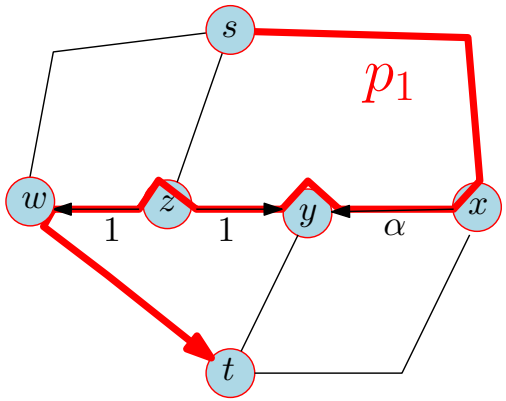
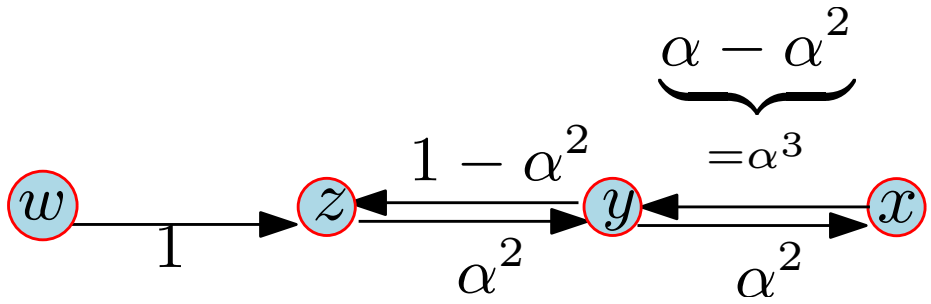
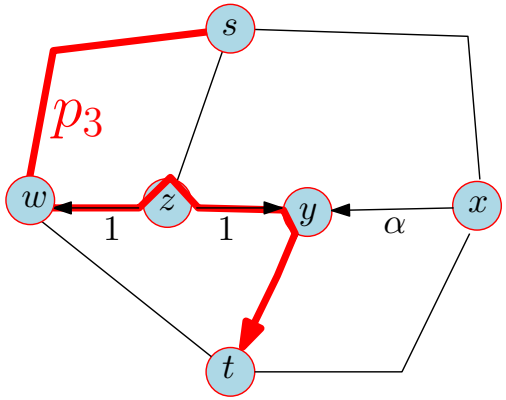
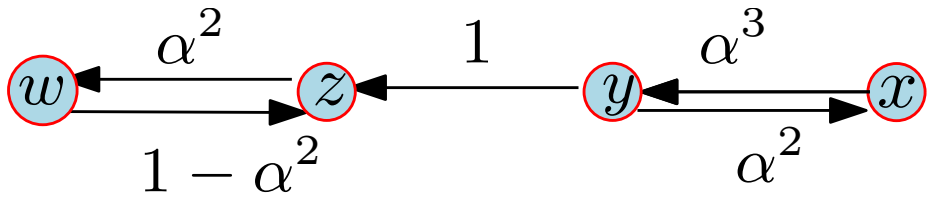
Let it flow II

#	Augment. path π	c_π	New residual network
1.		α	
2.		α	

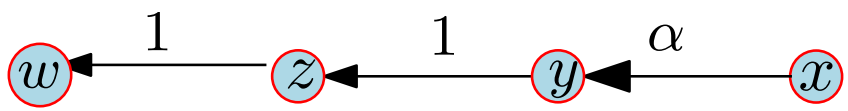
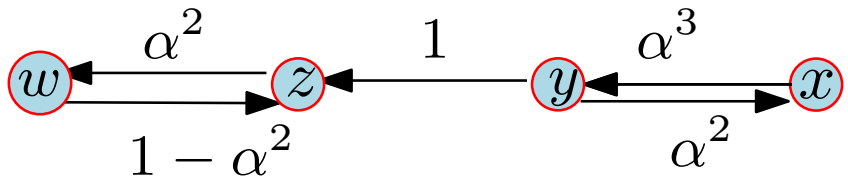
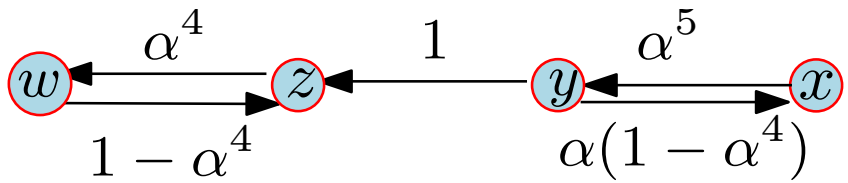
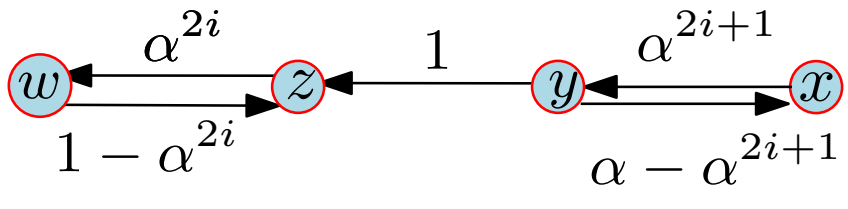
Let it flow II

2.		α^2	
3.		α^2	

Let it flow III

3.		α^2	
4.		α^2	

Let it flow III

moves	Residual network after
0	
moves **0, (1, 2, 3, 4)**	
moves **0, (1, 2, 3, 4)²**	
0.(1, 2, 3, 4)^i	

Namely, the algorithm never terminates.

Notes

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Notes

E. A. Dinic. Algorithm for solution of a problem of maximum flow in a network with power estimation. *Soviet Math. Doklady*, 11: 1277–1280, 1970.

J. Edmonds and R. M. Karp. Theoretical improvements in algorithmic efficiency for network flow problems. *J. Assoc. Comput. Mach.*, 19(2):248–264, 1972.