Network Flows

Lecture 17
March 19, 2015

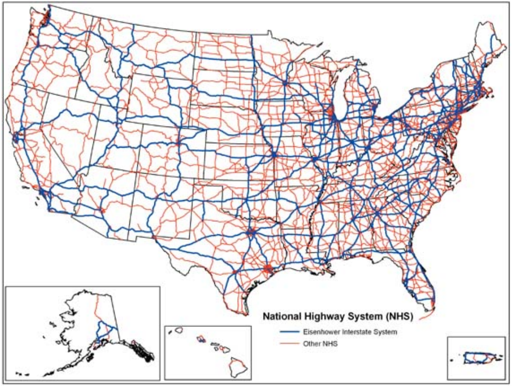
Everything flows

Panta rei – everything flows (literally).
Heraclitus (535–475 BC)

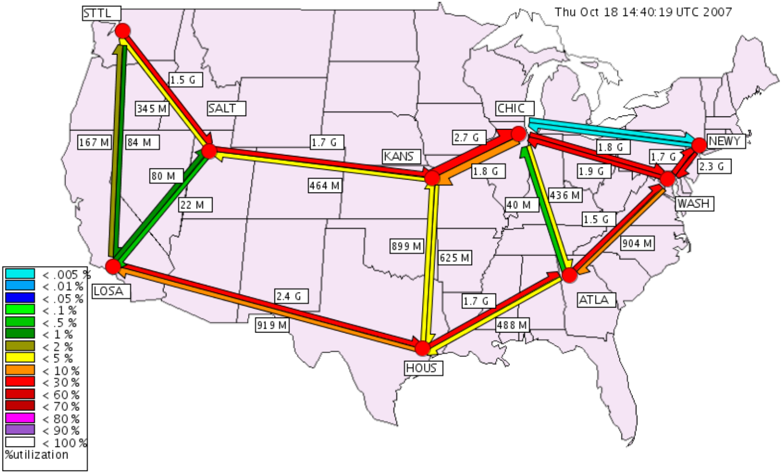
Part I

Network Flows: Introduction and Setup

Transportation/Road Network



Internet Backbone Network



Common Features of Flow Networks

1. **Network** represented by a (directed) *graph* $\mathbf{G = (V, E)}$.
2. Each edge $\mathbf{e}$ has a **capacity** $\mathbf{c(e) \geq 0}$ that limits amount of *traffic* on $\mathbf{e}$.
3. *Source(s)* of traffic/data.
4. *Sink(s)* of traffic/data.
5. Traffic *flows* from sources to sinks.
6. Traffic is *switched/interchanged* at nodes.
7. **Flow** abstract term to indicate stuff (traffic/data/etc) that **flows** from sources to sinks.

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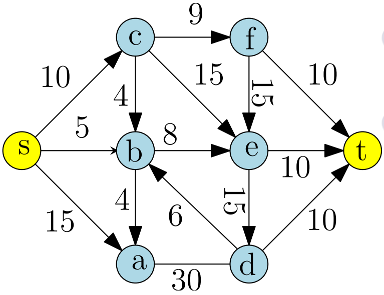
Single Source/Single Sink Flows

Simple setting:

1. Single source s and single sink t.
2. Every other node v is an **internal** node.
3. Flow originates at s and terminates at t.



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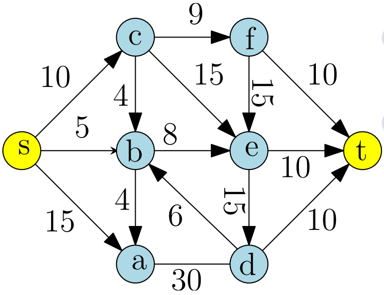

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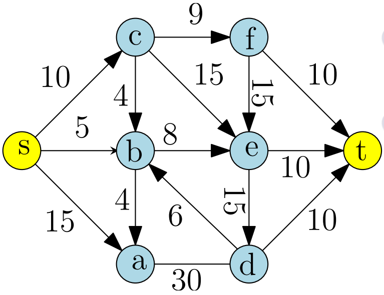

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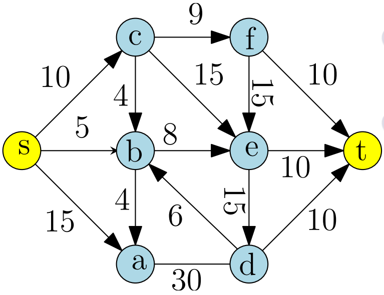

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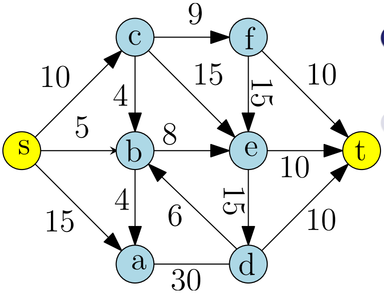

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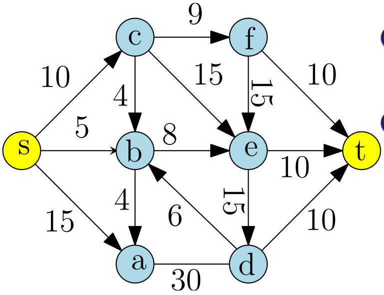

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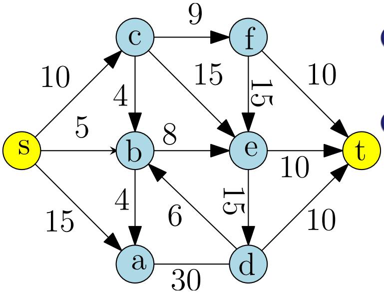

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Edge Based Definition of Flow

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Flow in network $G = (V, E)$, is function $f : E \to \mathbb{R}^{\geq 0}$ s.t.

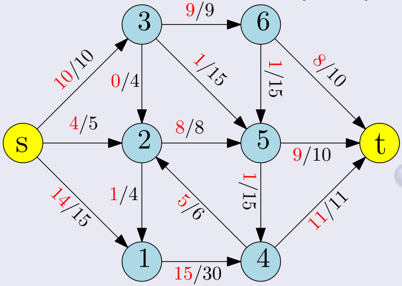

Figure: Flow with value.

1. Capacity Constraint: For each edge e, $f(e) \leq c(e)$.

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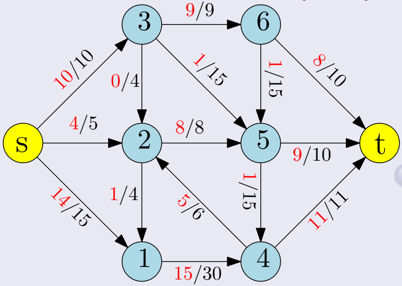

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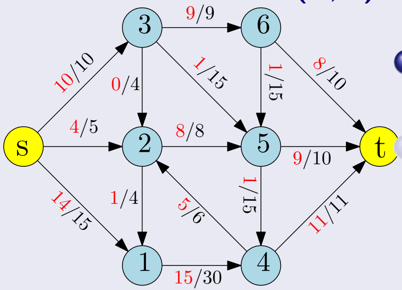

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$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

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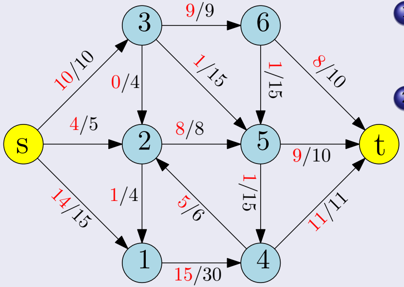

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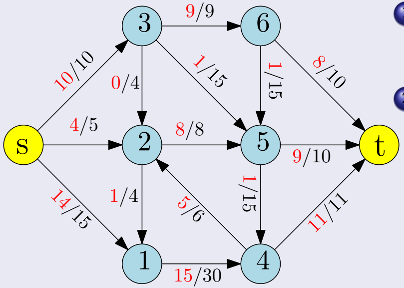

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Flow...

Conservation of flow law is also known as **Kirchhoff's law**.

More Definitions and Notation

Notation

1. The inflow into a vertex v is $f^{\text{in}}(v) = \sum_{e \text{ into } v} f(e)$ and the outflow is $f^{\text{out}}(v) = \sum_{e \text{ out of } v} f(e)$
2. For a set of vertices A, $f^{\text{in}}(A) = \sum_{e \text{ into } A} f(e)$. Outflow $f^{\text{out}}(A)$ is defined analogously

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For a network $G = (V, E)$ with source s, the **value** of flow f is defined as $v(f) = f^{\text{out}}(s) - f^{\text{in}}(s)$.

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A Path Based Definition of Flow

Intuition: Flow goes from source s to sink t along a path.

$\mathcal{P}$: set of all paths from s to t. $|\mathcal{P}|$ can be **exponential** in n.

Definition (Flow by paths.)

A **flow** in network $G = (V, E)$, is function $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ s.t.

1. Capacity Constraint: For each edge e, total flow on e is $\leq c(e)$.

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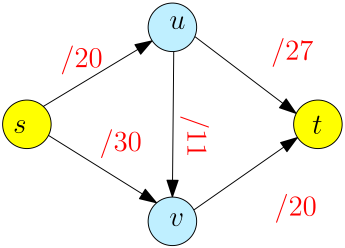

$\mathcal{P} = \{p_1, p_2, p_3\}$
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$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$

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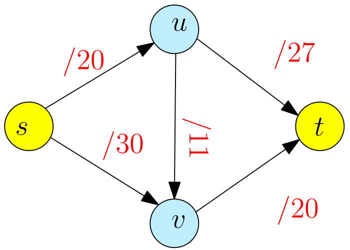

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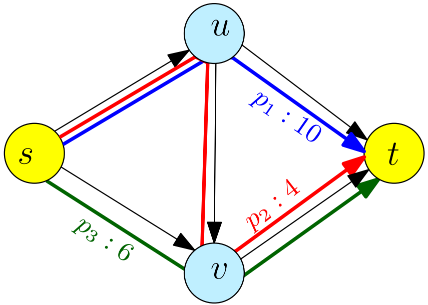

Path based flow implies edge based flow

Lemma

Given a path based flow $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ *there is an edge based flow* $f' : E \rightarrow \mathbb{R}^{\geq 0}$ *of the same value.*

Proof.

For each edge e define $f'(e) = \sum_{p: e \in p} f(p)$.

Exercise: Verify capacity and conservation constraints for f'.

Exercise: Verify that value of f and f' are equal □

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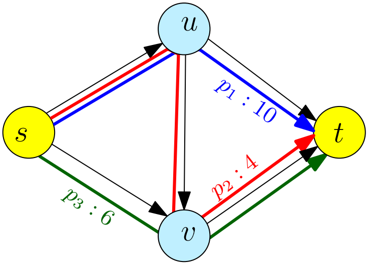

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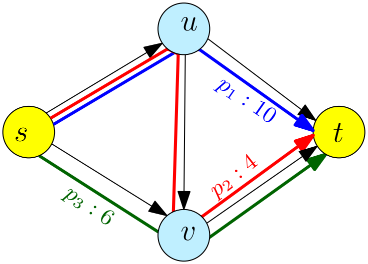

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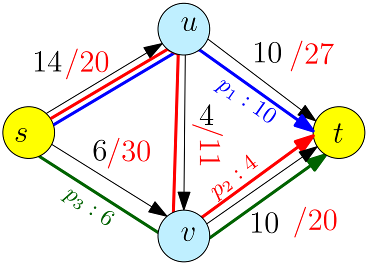

$f'(s \to u) = 14$
$f'(u \to v) = 4$
$f'(s \to v) = 6$
$f'(u \to t) = 10$
$f'(v \to t) = 10$

Flow Decomposition

Edge based flow to Path based Flow

Lemma

Given an edge based flow $f_1 : E \rightarrow \mathbb{R}^{\geq 0}$, there is a path based flow $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ of same value. Moreover, f assigns non-negative flow to at most m paths where $|E| = m$ and $|V| = n$. Given f_1, the path based flow can be computed in $O(mn)$ time.

Flow Decomposition

Edge based flow to Path based Flow

Proof Idea.

1. Remove all edges with $f_1(e) = 0$.
2. Find a path p from s to t.
3. Assign $f(p)$ to be $\min_{e \in p} f_1(e)$.
4. Reduce $f_1(e)$ for all $e \in p$ by $f(p)$.
5. Repeat until no path from s to t.
6. In each iteration at least on edge has flow reduced to zero.
7. Hence, at most m iterations. Can be implemented in $O(m(m+n))$ time. $O(mn)$ time requires care. □

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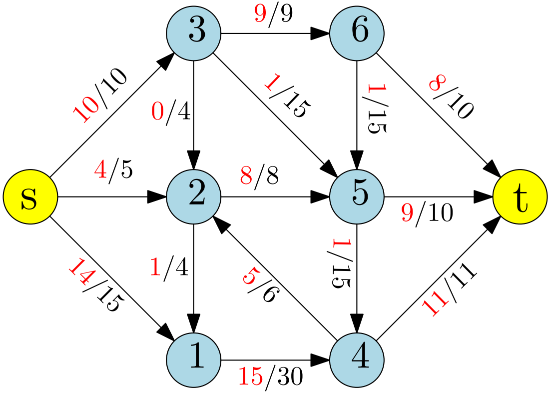

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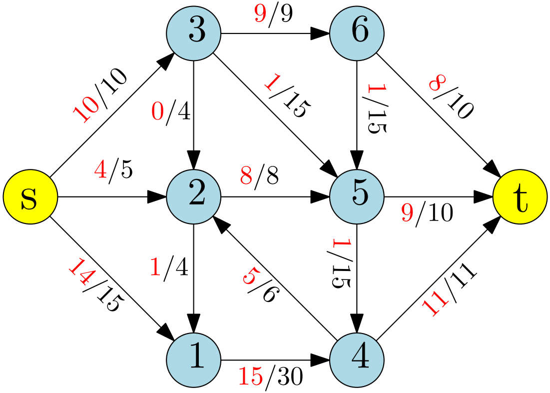

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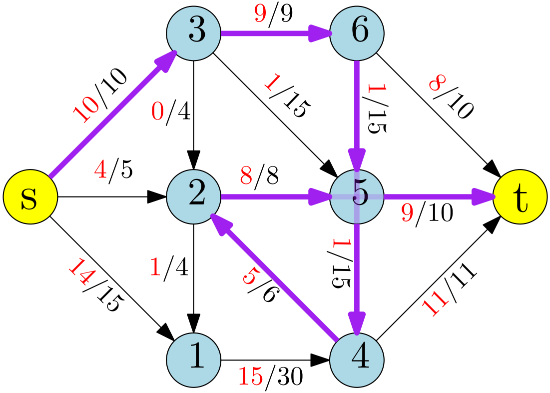

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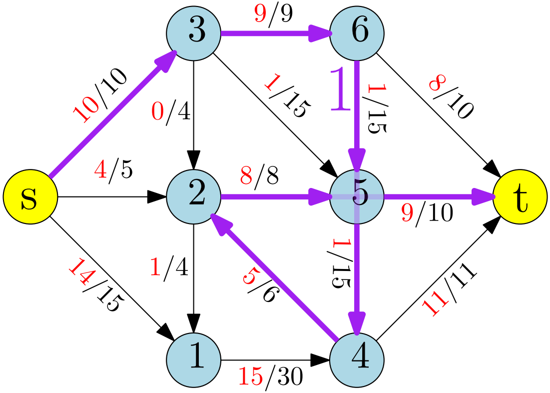

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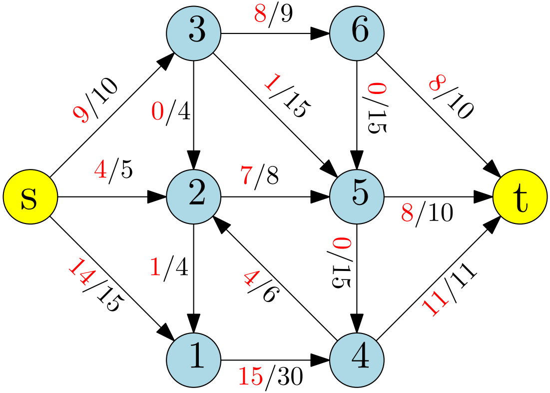

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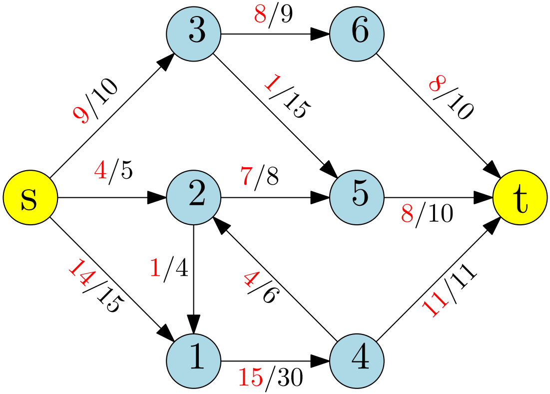

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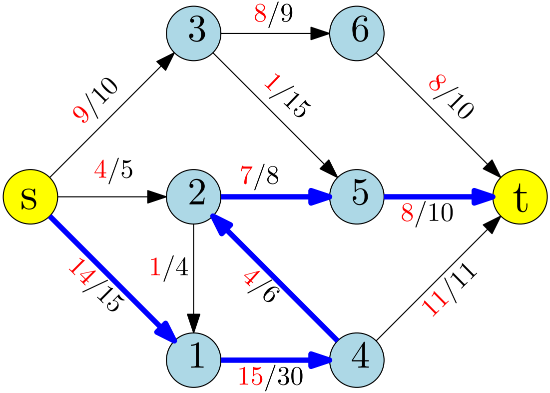

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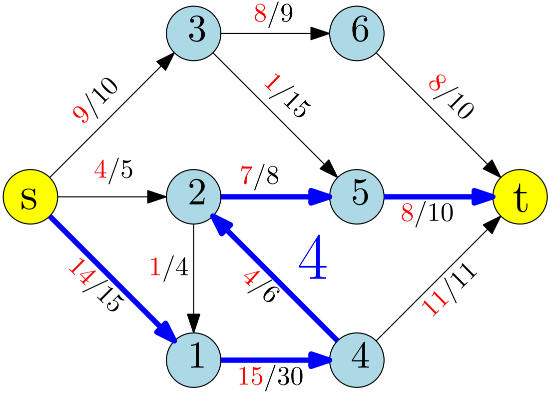

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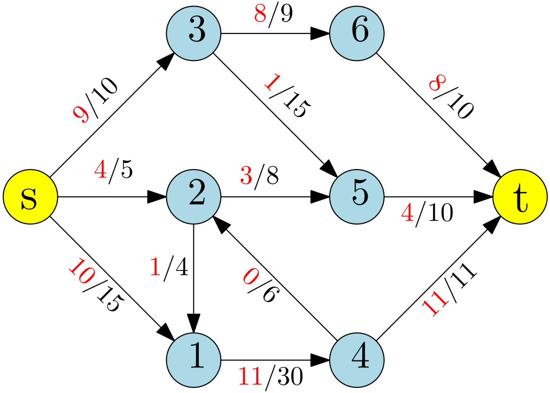

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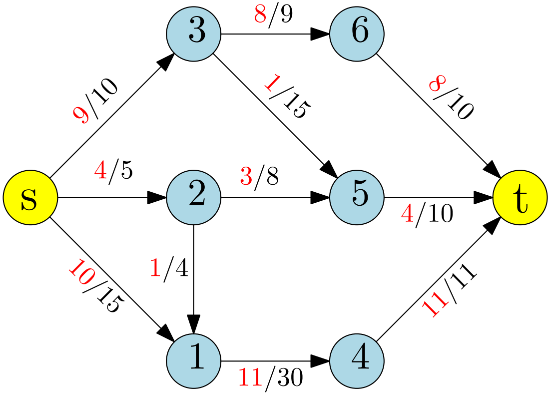

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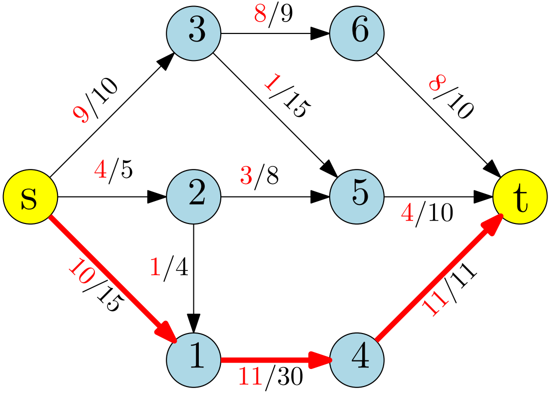

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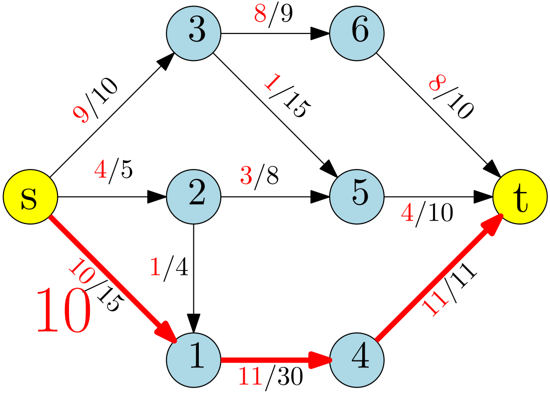

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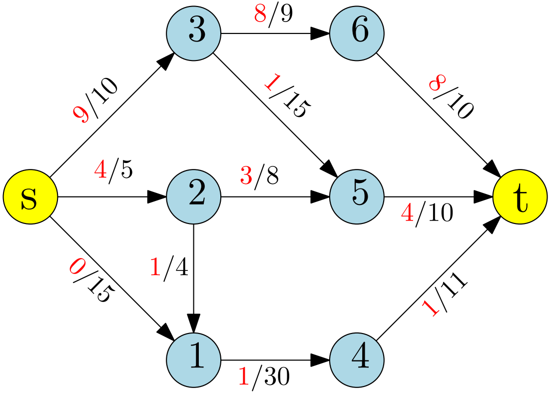

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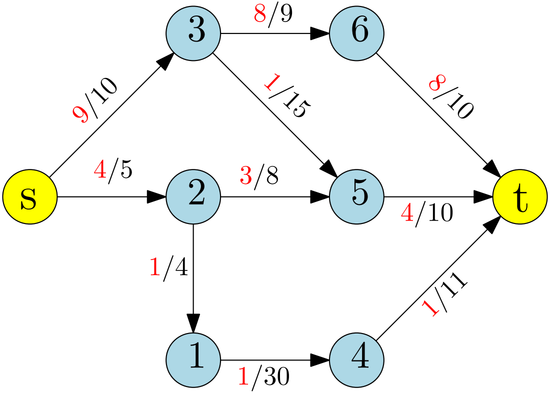

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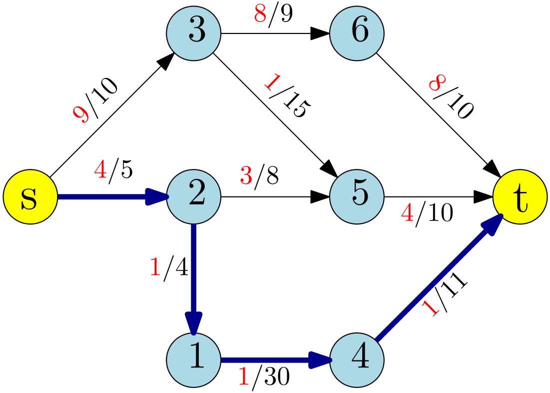

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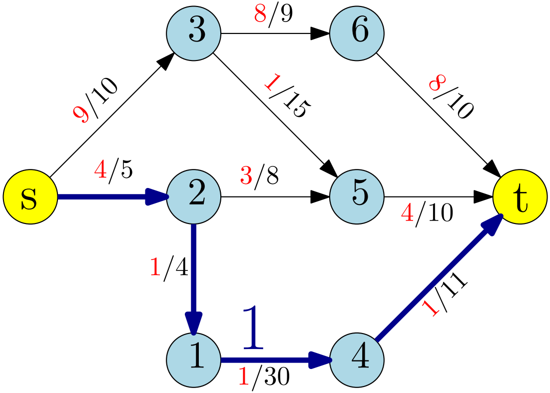

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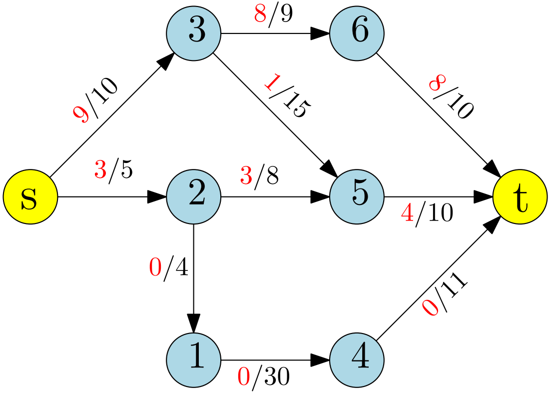

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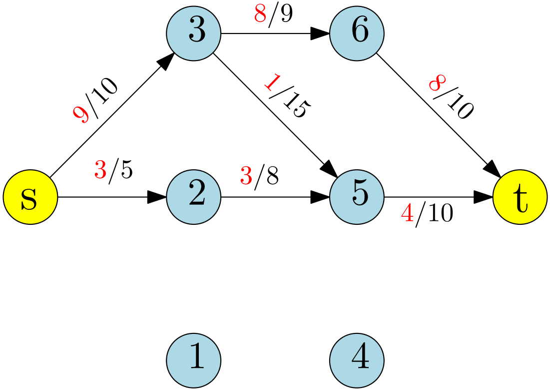

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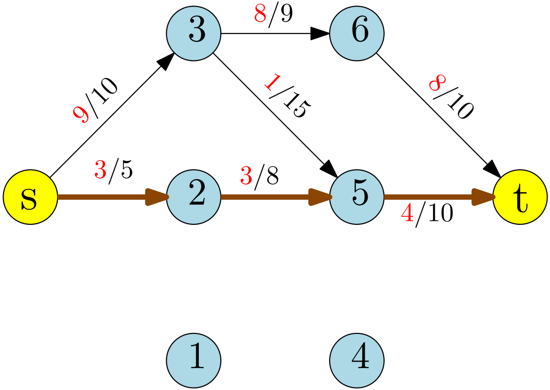

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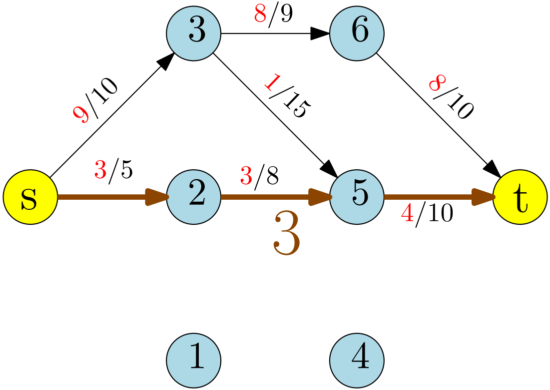

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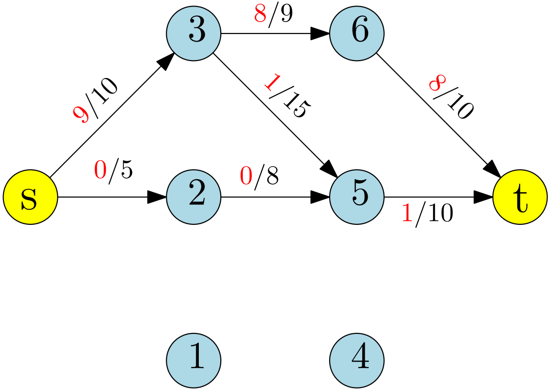

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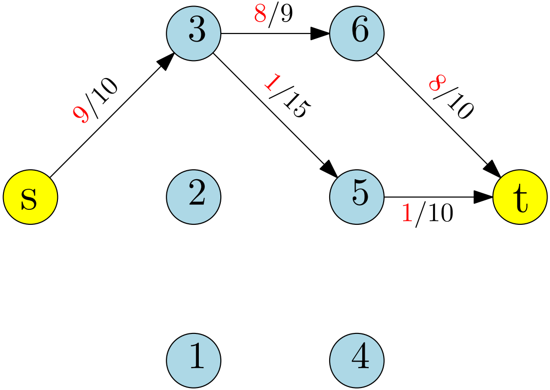

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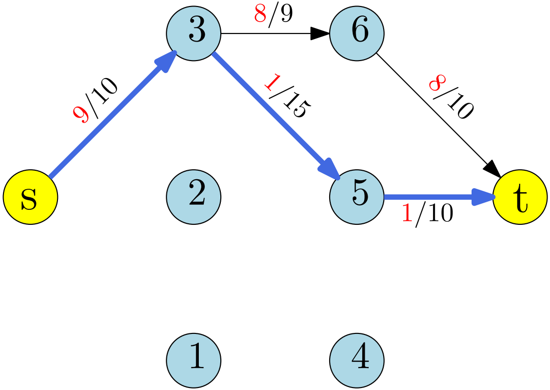

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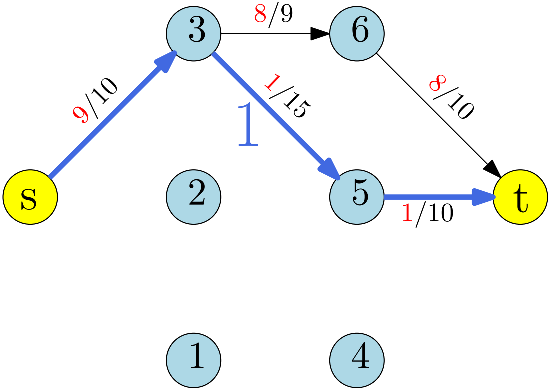

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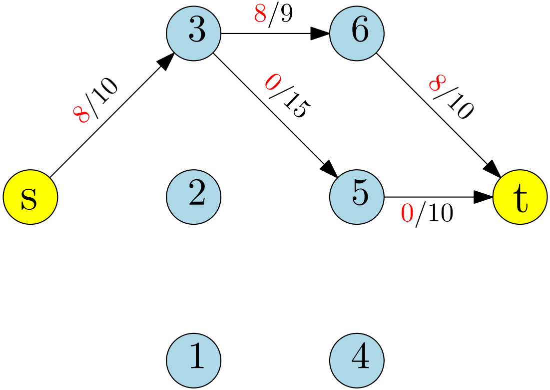

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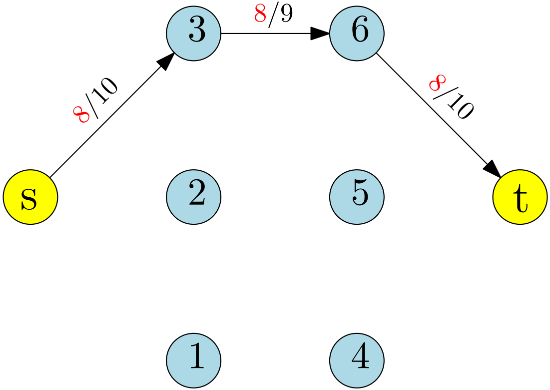

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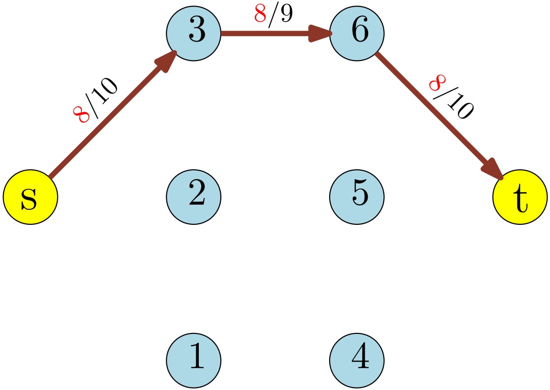

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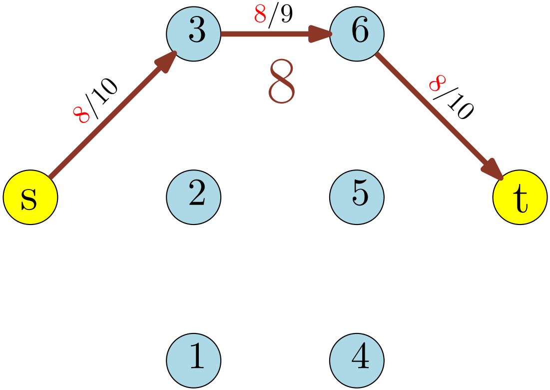

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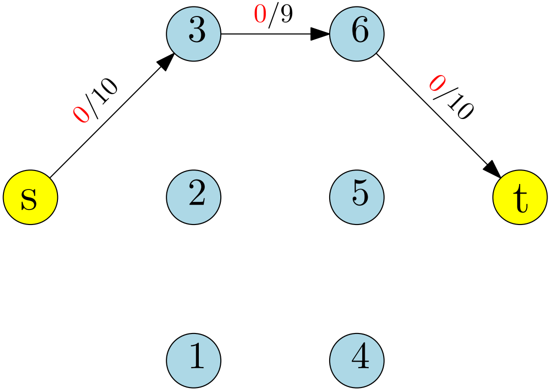

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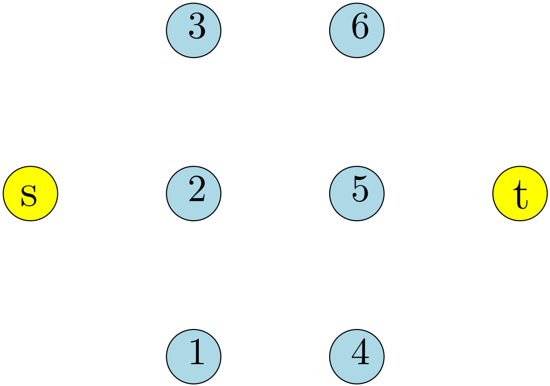

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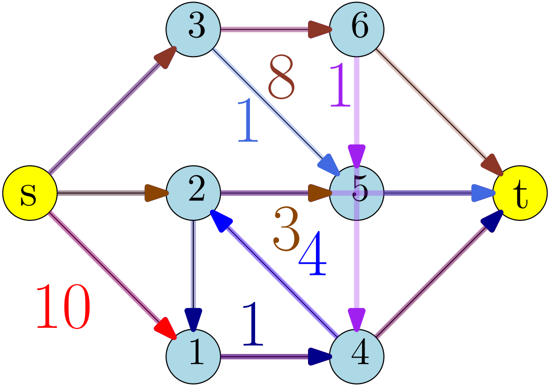
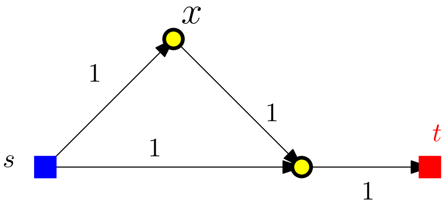


Path flow decomposition

Do not have to be efficient...



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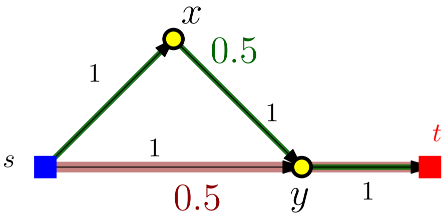

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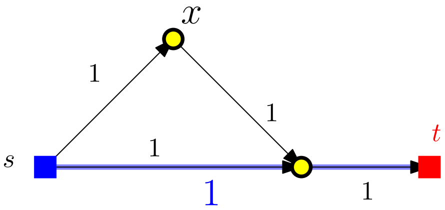

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   1. **compact** representation, only m values to be specified, and
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   1. in some applications, paths more natural,
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1. The **network flow problem**:

Problem

Input: A network G with capacity c and source s and sink t.

Goal: Find flow of **maximum** value.

2. Question: Given a flow network, what is an *upper bound* on the maximum flow between source and sink?

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Given a flow network an **s-t cut** is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \rightarrow t$ path in $E - E'$.
The **capacity** of a cut E' is $c(E') = \sum_{e \in E'} c(e)$.

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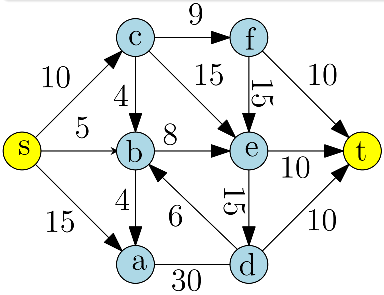

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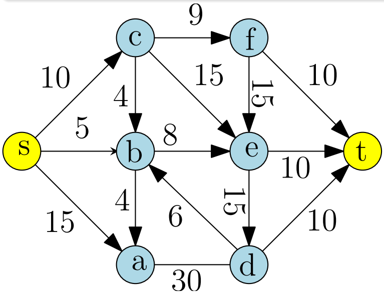

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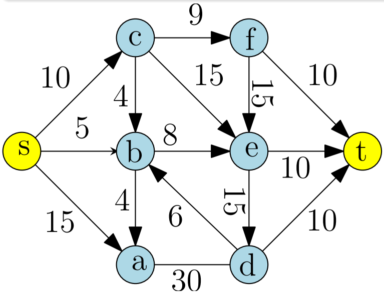

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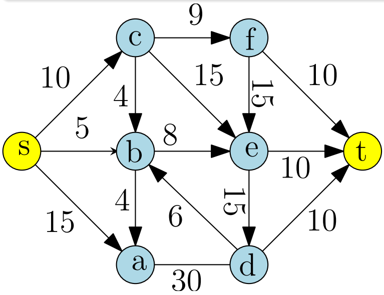

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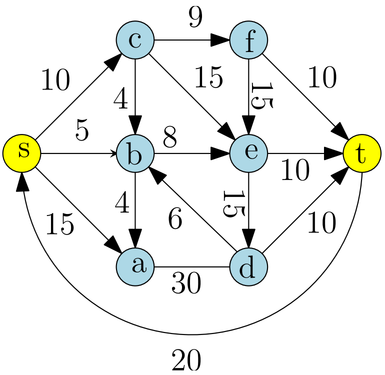
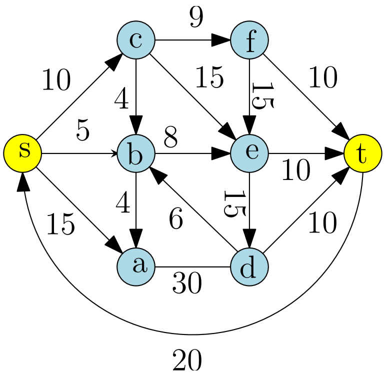
s — t cuts

A death by a thousand cuts

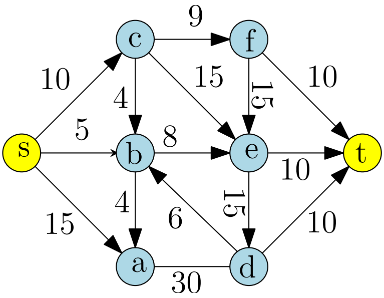




Minimal Cut

Definition (Minimal s-t cut.)

Given a s-t flow network $G = (V, E)$, $E' \subseteq E$ is a **minimal cut** if for all $e \in E'$, if $E' \setminus \{e\}$ is not a cut.



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Minimal Cut

Definition (Minimal s-t cut.)

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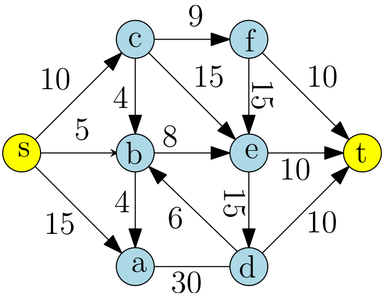

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Cuts as Vertex Partitions

1. Let $A \subset V$ such that
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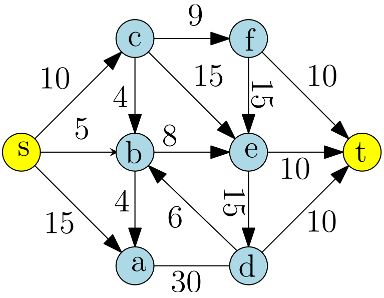
2. The **cut** (A, B) is the set of edges $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$. Cut (A, B) is set of edges leaving A.



Claim

(A, B) is an s-t cut.

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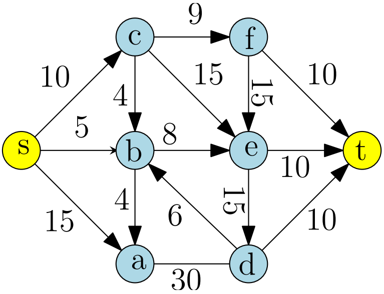

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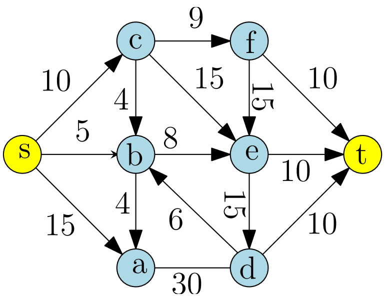

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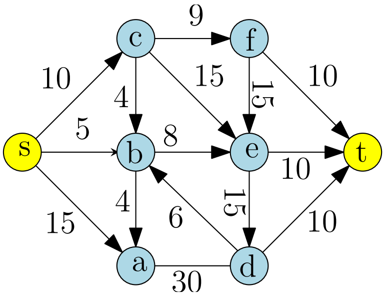

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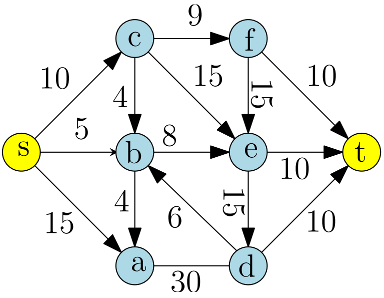

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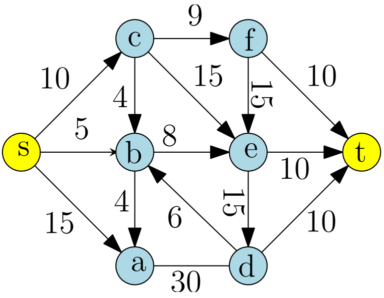

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Suppose E' *is an* s-t *cut. Then there is a cut* (A, B) *such that* $(A, B) \subseteq E'$.

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Every minimal s-t *cut* E' *is a cut of the form* (A, B).

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Definition

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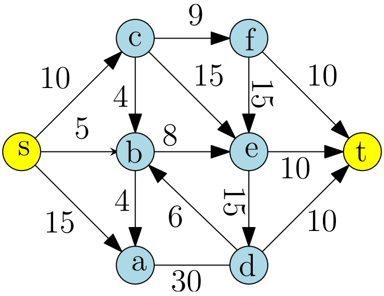

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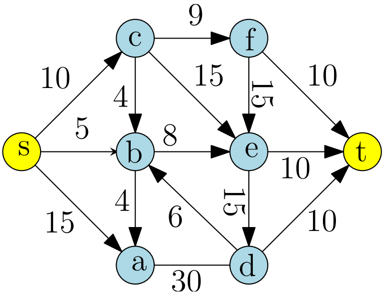

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Input A flow network G

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Flows and Cuts

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For any s-t *cut* E', **maximum** s-t *flow* $\leq$ *capacity of* E'.

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Max-Flow Min-Cut Theorem

Theorem

In any flow network:

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Input A network G with capacity c and source s and sink t.

Goal Find flow of **maximum** value from s to t.

Exercise: Given G, s, t as above, show that one can remove all edges into s and all edges out of t without affecting the flow value between s and t.

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Dinic, E. A. (1970). Algorithm for solution of a problem of maximum flow in a network with power estimation. *Soviet Math. Doklady*, 11:1277–1280.

Edmonds, J. and Karp, R. M. (1972). Theoretical improvements in algorithmic efficiency for network flow problems. *J. Assoc. Comput. Mach.*, 19(2):248–264.