OLD CS 473: Fundamental Algorithms, Spring 2015

Introduction to Randomized Algorithms: QuickSort and QuickSelect

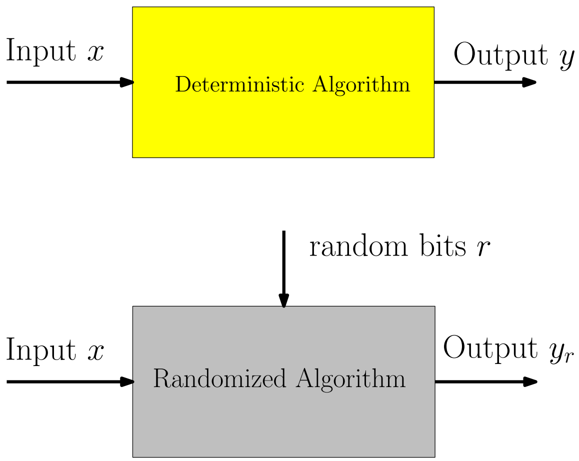
Lecture 14

March 10, 2015

Part I

Introduction to Randomized Algorithms

Randomized Algorithms



Example: Randomized **QuickSort**

QuickSort Hoare [1962]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

Randomized **QuickSort**

1. Pick a pivot element *uniformly at random* from the array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

Example: Randomized Quicksort

Recall: **QuickSort** can take $\Omega(n^2)$ time to sort array of size n.

Theorem

Randomized **QuickSort** *sorts a given array of length n in $O(n \log n)$ expected time.*

Note: On *every* input randomized **QuickSort** takes $O(n \log n)$ time in expectation. On *every* input it may take $\Omega(n^2)$ time with some small probability.

Example: Verifying Matrix Multiplication

Problem

Given three $n \times n$ matrices A, B, C is $AB = C$?

Deterministic algorithm:

1. Multiply A and B and check if equal to C.
2. Running time? $O(n^3)$ by straight forward approach. $O(n^{2.37})$ with fast matrix multiplication (complicated and impractical).

Example: Verifying Matrix Multiplication

Problem

Given three $n \times n$ matrices A, B, C is $AB = C$?

Randomized algorithm:

1. Pick a random $n \times 1$ vector r.
2. Return the answer of the equality $ABr = Cr$.
3. Running time? $O(n^2)$!

Theorem

If $AB = C$ then the algorithm will always say YES. If $AB \neq C$ then the algorithm will say YES with probability at most $1/2$. Can repeat the algorithm 100 times independently to reduce the probability of a false positive to $1/2^{100}$.

Why randomized algorithms?

1. Many applications: algorithms, data structures and CS.
2. In some cases only known algorithms are randomized or randomness is provably necessary.
3. Often randomized algorithms are (much) simpler and/or more efficient.
4. Several deep connections to mathematics, physics etc.
5. ...
6. Lots of fun!

Where do I get random bits?

Question: Are true random bits available in practice?

1. Buy them!
2. CPUs use physical phenomena to generate random bits.
3. Can use pseudo-random bits or semi-random bits from nature. Several fundamental unresolved questions in complexity theory on this topic. Beyond the scope of this course.
4. In practice pseudo-random generators work quite well in many applications.
5. The model is interesting to think in the abstract and is very useful even as a theoretical construct. One can *derandomize* randomized algorithms to obtain deterministic algorithms.

Average case analysis vs Randomized algorithms

Average case analysis:

1. Fix a deterministic algorithm.
2. Assume inputs comes from a probability distribution.
3. Analyze the algorithm's *average* performance over the distribution over inputs.

Randomized algorithms:

1. Algorithm uses random bits in addition to input.
2. Analyze algorithms *average* performance over the given input where the average is over the random bits that the algorithm uses.
3. On each input behaviour of algorithm is random. Analyze worst-case over all inputs of the (average) performance.

Discrete Probability

We restrict attention to finite probability spaces.

Definition

A discrete probability space is a pair $(\Omega, \Pr)$ consists of finite set Ω of **elementary events** and function $p : \Omega \to [0, 1]$ which assigns a probability $\Pr[\omega]$ for each $\omega \in \Omega$ such that $\sum_{\omega \in \Omega} \Pr[\omega] = 1$.

Example

An unbiased coin. $\Omega = \{H, T\}$ and $\Pr[H] = \Pr[T] = 1/2$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for $1 \leq i \leq 6$.

Discrete Probability

And more examples

Example

A biased coin. $\Omega = \{H, T\}$ and $\Pr[H] = 2/3, \Pr[T] = 1/3$.

Example

Two independent unbiased coins. $\Omega = \{HH, TT, HT, TH\}$ and $\Pr[HH] = \Pr[TT] = \Pr[HT] = \Pr[TH] = 1/4$.

Example

A pair of (highly) correlated dice.
$\Omega = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$.
$\Pr[i, i] = 1/6$ for $1 \leq i \leq 6$ and $\Pr[i, j] = 0$ if $i \neq j$.

Events

Definition

Given a probability space $(\Omega, \Pr)$ an **event** is a subset of Ω. In other words an event is a collection of elementary events. The probability of an event A, denoted by $\Pr[A]$, is $\sum_{\omega \in A} \Pr[\omega]$.

Definition

The **complement event** of an event $A \subseteq \Omega$ is the event $\Omega \setminus A$ frequently denoted by $\bar{A}$.

Events

Examples

Example

A pair of independent dice. $\Omega = \{(i,j) \mid 1 \le i \le 6, 1 \le j \le 6\}$.

1. Let A be the event that the sum of the two numbers on the dice is even.
   Then $A = \left\{ (i,j) \in \Omega \;\middle|\; (i+j) \text{ is even} \right\}$.
   $\Pr[A] = |A|/36 = 1/2$.
2. Let B be the event that the first die has 1. Then
   $B = \left\{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \right\}$.
   $\Pr[B] = 6/36 = 1/6$.

Independent Events

Definition

Given a probability space $(\Omega, \Pr)$ and two events A, B are **independent** if and only if $\Pr[A \cap B] = \Pr[A] \Pr[B]$. Otherwise they are *dependent*. In other words A, B independent implies one does not affect the other.

Example

Two coins. $\Omega = \{HH, TT, HT, TH\}$ and
$\Pr[HH] = \Pr[TT] = \Pr[HT] = \Pr[TH] = 1/4$.

1. A is the event that the first coin is heads and B is the event that second coin is tails. A, B are independent.
2. A is the event that the two coins are different. B is the event that the second coin is heads. A, B independent.

Independent Events

Examples

Example

A is the event that both are not tails and B is event that second coin is heads. A, B are dependent.

Union bound

The probability of the union of two events, is $\leq$ the probability of the sum of their probabilities.

Lemma

For any two events $\mathcal{E}$ and $\mathcal{F}$, we have that
$\mathbf{Pr}\left[\mathcal{E} \cup \mathcal{F}\right] \leq \mathbf{Pr}\left[\mathcal{E}\right] + \mathbf{Pr}\left[\mathcal{F}\right]$.

Proof.

Consider $\mathcal{E}$ and $\mathcal{F}$ to be a collection of elmentery events (which they are). We have

$$\mathbf{Pr}\left[\mathcal{E} \cup \mathcal{F}\right] = \sum_{x \in \mathcal{E} \cup \mathcal{F}} \mathbf{Pr}[x]$$
$$\leq \sum_{x \in \mathcal{E}} \mathbf{Pr}[x] + \sum_{x \in \mathcal{F}} \mathbf{Pr}[x] = \mathbf{Pr}\left[\mathcal{E}\right] + \mathbf{Pr}\left[\mathcal{F}\right].$$

Random Variables

Definition

Given a probability space $(\Omega, \mathbf{Pr})$ a (real-valued) random variable X over Ω is a function that maps each elementary event to a real number. In other words $X : \Omega \rightarrow \mathbb{R}$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbf{Pr}[i] = 1/6$ for $1 \leq i \leq 6$.

1. $X : \Omega \rightarrow \mathbb{R}$ where $X(i) = i \mod 2$.
2. $Y : \Omega \rightarrow \mathbb{R}$ where $Y(i) = i^2$.

Definition

A **binary random variable** is one that takes on values in $\{0, 1\}$.

Indicator Random Variables

Special type of random variables that are quite useful.

Definition

Given a probability space $(\Omega, \mathbf{Pr})$ and an event $A \subseteq \Omega$ the indicator random variable X_A is a binary random variable where $X_A(\omega) = 1$ if $\omega \in A$ and $X_A(\omega) = 0$ if $\omega \notin A$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbf{Pr}[i] = 1/6$ for $1 \leq i \leq 6$. Let A be the even that i is divisible by 3. Then $X_A(i) = 1$ if $i = 3, 6$ and 0 otherwise.

Expectation

Definition

For a random variable X over a probability space $(\Omega, \mathbf{Pr})$ the **expectation** of X is defined as $\sum_{\omega \in \Omega} \mathbf{Pr}[\omega]\, X(\omega)$. In other words, the expectation is the average value of X according to the probabilities given by $\mathbf{Pr}[\cdot]$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbf{Pr}[i] = 1/6$ for $1 \leq i \leq 6$.

1. $X : \Omega \rightarrow \mathbb{R}$ where $X(i) = i \mod 2$. Then $\mathbf{E}[X] = 1/2$.
2. $Y : \Omega \rightarrow \mathbb{R}$ where $Y(i) = i^2$. Then $\mathbf{E}[Y] = \sum_{i=1}^{6} \frac{1}{6} \cdot i^2 = 91/6$.

Expectation

Proposition

For an indicator variable X_A, $\mathbf{E}[X_A] = \Pr[A]$.

Proof.

$$\begin{aligned}\mathbf{E}[X_A] &= \sum_{y\in\Omega} X_A(y)\Pr[y]\\ &= \sum_{y\in A} 1\cdot\Pr[y] + \sum_{y\in\Omega\setminus A} 0\cdot\Pr[y]\\ &= \sum_{y\in A}\Pr[y]\\ &= \Pr[A].\end{aligned}$$

□

Linearity of Expectation

Lemma

Let X, Y *be two random variables (not necessarily independent) over a probability space* $(\Omega, \Pr)$. *Then* $\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$.

Proof.

$$\begin{aligned}\mathbf{E}[X+Y] &= \sum_{\omega\in\Omega}\Pr[\omega]\,(X(\omega)+Y(\omega))\\ &= \sum_{\omega\in\Omega}\Pr[\omega]\,X(\omega) + \sum_{\omega\in\Omega}\Pr[\omega]\,Y(\omega) = \mathbf{E}[X]+\mathbf{E}[Y].\end{aligned}$$

□

Corollary

$\mathbf{E}[a_1X_1 + a_2X_2 + \ldots + a_nX_n] = \sum_{i=1}^{n} a_i\,\mathbf{E}[X_i]$.

Types of Randomized Algorithms

Typically one encounters the following types:

1. **Las Vegas randomized algorithms:** for a given input x output of algorithm is always correct but the running time is a random variable. In this case we are interested in analyzing the *expected* running time.
2. **Monte Carlo randomized algorithms:** for a given input x the running time is deterministic but the output is random; correct with some probability. In this case we are interested in analyzing the *probability* of the correct output (and also the running time).
3. Algorithms whose running time and output may both be random variables.

Analyzing Las Vegas Algorithms

Deterministic algorithm Q for a problem Π:

1. Let $Q(x)$ be the time for Q to run on input x of length $|x|$.
2. Worst-case analysis: run time on worst input for a given size n.

$$T_{wc}(n) = \max_{x:|x|=n} Q(x).$$

Randomized algorithm R for a problem Π:

1. Let $R(x)$ be the time for Q to run on input x of length $|x|$.
2. $R(x)$ is a random variable: depends on random bits used by R.
3. $\mathbf{E}[R(x)]$ is the expected running time for R on x
4. Worst-case analysis: expected time on worst input of size n

$$T_{rand-wc}(n) = \max_{x:|x|=n} \mathbf{E}[Q(x)].$$

Analyzing Monte Carlo Algorithms

Randomized algorithm M for a problem Π:

1. Let $M(x)$ be the time for M to run on input x of length $|x|$. For Monte Carlo, assumption is that run time is deterministic.
2. Let $\Pr[x]$ be the probability that M is correct on x.
3. $\Pr[x]$ is a random variable: depends on random bits used by M.
4. Worst-case analysis: success probability on worst input

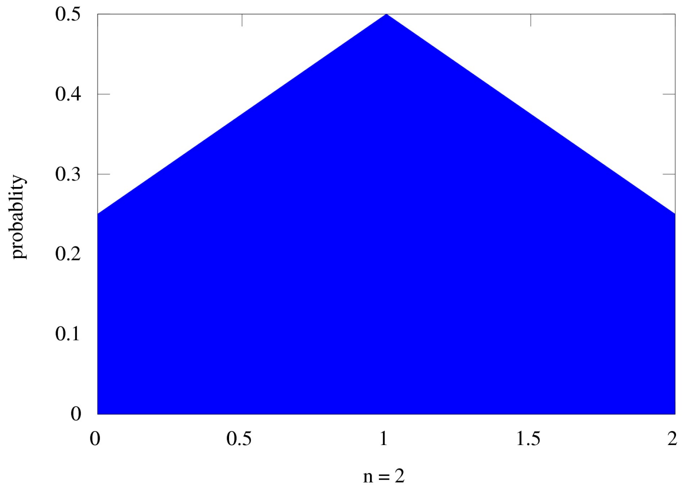
$$P_{rand-wc}(n) = \min_{x:|x|=n} \Pr[x].$$

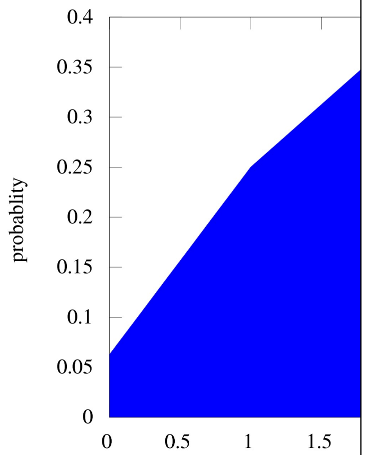
Part II

Why does randomization help?

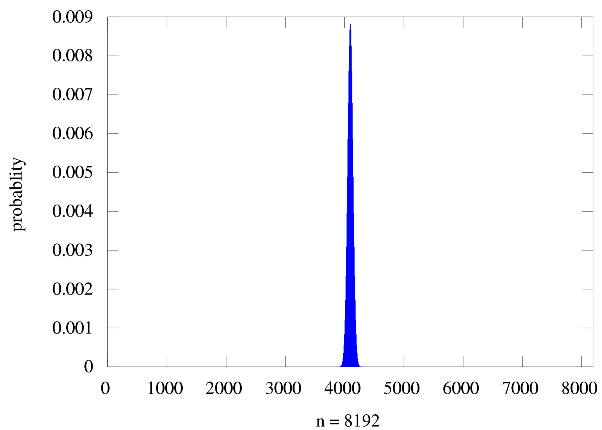
Massive randomness.. Is not that random.

Consider flipping a fair coin n times independently, head given **1**, tail gives zero. How many heads? ...we get a binomial distribution.





Massive randomness.. Is not that random.



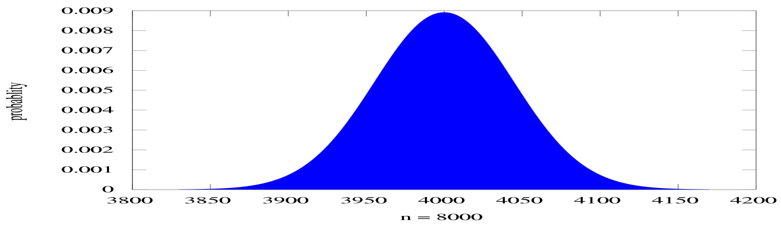
This is known as **concentration of mass**.
This is a very special case of the **law of large numbers**.

Side note...

Law of large numbers (weakest form)...

Informal statement of law of large numbers

For n large enough, the middle portion of the binomial distribution looks like (converges to) the normal/Gaussian distribution.



Massive randomness.. Is not that random.

Intuitive conclusion

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.

Binomial distribution

X_n = numbers of heads when flipping a coin n times.

Claim

$\mathbf{Pr}\Big[X_n = i\Big] = \frac{\binom{n}{i}}{2^n}$.

Where: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
Indeed, $\binom{n}{i}$ is the number of ways to choose i elements out of n elements (i.e., pick which i coin flip come up heads).
Each specific such possibility (say $0100010...$) had probability $1/2^n$.
We are interested in the bad event $\mathbf{Pr}[X_n \leq n/4]$ (way too few heads). We are going to prove this probability is tiny.

Binomial distribution

Playing around with binomial coefficients

Lemma

$n! \geq (n/e)^n$.

Proof.

$$\frac{n^n}{n!} \leq \sum_{i=0}^{\infty} \frac{n^i}{i!} = e^n,$$

by the Taylor expansion of $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$. This implies that $(n/e)^n \leq n!$, as required. □

Binomial distribution

Playing around with binomial coefficients

Lemma

For any $k \leq n$, *we have* $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$.

Proof.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$
$$\leq \frac{n^k}{k!} \leq \frac{n^k}{\left(\frac{k}{e}\right)^k} = \left(\frac{ne}{k}\right)^k.$$

since $k! \geq (k/e)^k$ (by previous lemma). □

Binomial distribution

Playing around with binomial coefficients

$$\Pr\left[X_n \leq \frac{n}{4}\right] = \sum_{k=0}^{n/4} \frac{1}{2^n}\binom{n}{k} = \frac{1}{2^n}\sum_{k=0}^{n/4}\binom{n}{k} \leq \frac{1}{2^n} 2 \cdot \binom{n}{n/4}$$

For $k \leq n/4$ the above sequence behave like a geometric variable.

$$\binom{n}{k+1} \Big/ \binom{n}{k} = \frac{n!}{(k+1)!(n-k-1)!} \Big/ \frac{n!}{(k)!(n-k)!}$$
$$= \frac{n-k}{k+1} \geq \frac{(3/4)n}{n/4+1} \geq 2.$$

Binomial distribution

Playing around with binomial coefficients

$$\Pr\left[X_n \leq \frac{n}{4}\right] \leq \frac{1}{2^n} 2 \cdot \binom{n}{n/4} \leq \frac{1}{2^n} 2 \cdot \left(\frac{ne}{n/4}\right)^{n/4} \leq 2 \cdot \left(\frac{4e}{2^4}\right)^{n/4}$$
$$\leq 2 \cdot 0.68^{n/4}.$$

We just proved the following theorem.

Theorem

Let X_n *be the random variable which is the number of heads when flipping an unbiased coin independently* n *times. Then*

$$\Pr\left[X_n \leq \frac{n}{4}\right] \leq 2 \cdot 0.68^{n/4} \text{ and } \Pr\left[X_n \geq \frac{3n}{4}\right] \leq 2 \cdot 0.68^{n/4}.$$

Part III

Randomized Quick Sort and Selection

Randomized QuickSort

Randomized QuickSort

1. Pick a pivot element *uniformly at random* from the array.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

Example

1. array: 16, 12, 14, 20, 5, 3, 18, 19, 1

Analysis via Recurrence

1. Given array A of size n, let $Q(A)$ be number of comparisons of randomized **QuickSort** on A.
2. Note that $Q(A)$ is a random variable.
3. Let A^i_{left} and A^i_{right} be the left and right arrays obtained if:

pivot is of rank i in A.

$$Q(A) = n + \sum_{i=1}^{n} \Pr\Big[\text{pivot has rank } i\Big] \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}})\right).$$

Since each element of A has probability exactly of $1/n$ of being chosen:

$$Q(A) = n + \sum_{i=1}^{n} \frac{1}{n} \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}})\right).$$

Analysis via Recurrence

Let $T(n) = \max_{A:|A|=n} \mathbf{E}[Q(A)]$ be the worst-case expected running time of randomized **QuickSort** on arrays of size n.

We have, for any A:

$$Q(A) = n + \sum_{i=1}^{n} \Pr\Big[\text{pivot has rank } i\Big] \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}})\right)$$

Therefore, by linearity of expectation:

$$\mathbf{E}\Big[Q(A)\Big] = n + \sum_{i=1}^{n} \Pr\left[\begin{array}{c}\text{pivot is}\\ \text{of rank } i\end{array}\right] \left(\mathbf{E}\Big[Q(A^i_{\text{left}})\Big] + \mathbf{E}\Big[Q(A^i_{\text{right}})\Big]\right).$$

$$\Rightarrow \quad \mathbf{E}\Big[Q(A)\Big] \le n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i)\right).$$

Analysis via Recurrence

Let $T(n) = \max_{A:|A|=n} \mathbf{E}[Q(A)]$ be the worst-case expected running time of randomized **QuickSort** on arrays of size n.
We derived:

$$\mathbf{E}\Big[Q(A)\Big] \leq n + \sum_{i=1}^{n} \frac{1}{n}\left(T(i-1) + T(n-i)\right).$$

Note that above holds for any A of size n. Therefore

$$\max_{A:|A|=n} \mathbf{E}[Q(A)] = T(n) \leq n + \sum_{i=1}^{n} \frac{1}{n}\left(T(i-1) + T(n-i)\right).$$

Solving the Recurrence

$$T(n) \leq n + \sum_{i=1}^{n} \frac{1}{n}\left(T(i-1) + T(n-i)\right)$$

with base case $T(1) = 0$.

Lemma

$T(n) = O(n \log n)$.

Proof.

(Guess and) Verify by induction. □

Hoare, C. A. R. (1962). Quicksort. *Comput. J.*, 5(1):10–15.