Breadth First Search, Dijkstra's Algorithm for Shortest Paths

Lecture 4

January 29, 2015

Part I

Breadth First Search

Breadth First Search (BFS)

Overview

(A) **BFS** is obtained from **BasicSearch** by processing edges using a **queue** data structure.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

Breadth First Search (BFS)

Overview

(A) **BFS** is obtained from **BasicSearch** by processing edges using a **queue** data structure.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

Breadth First Search (BFS)

Overview

(A) **BFS** is obtained from **BasicSearch** by processing edges using a **queue** data structure.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

Breadth First Search (BFS)

Overview

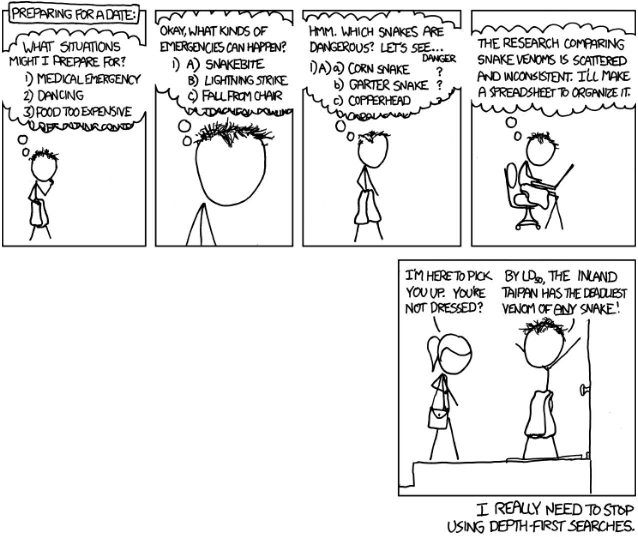
(A) **BFS** is obtained from **BasicSearch** by processing edges using a **queue** data structure.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

xkcd take on DFS



Queue Data Structure

Queues

queue: list of elements which supports the operations:

1. **enqueue**: Adds an element to the end of the list
2. **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.

BFS Algorithm

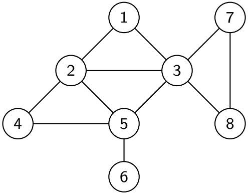
Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

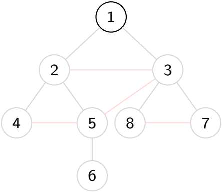
```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enq(v)
```

Proposition

BFS(s) *runs in* $O(n + m)$ *time.*

BFS: An Example in Undirected Graphs

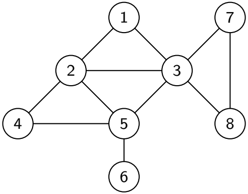


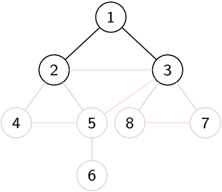


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

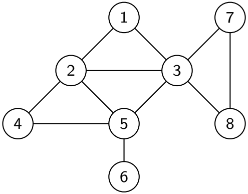


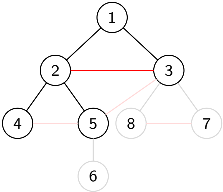


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

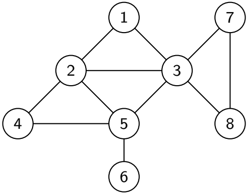


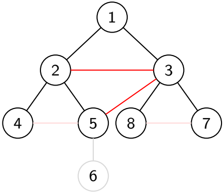


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

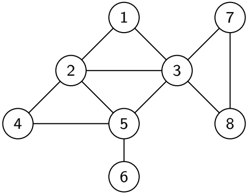


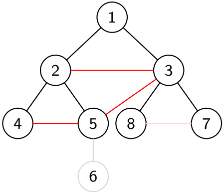


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

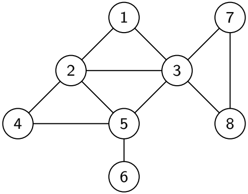


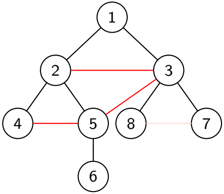


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

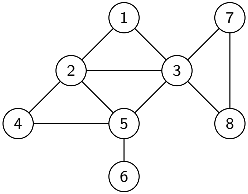


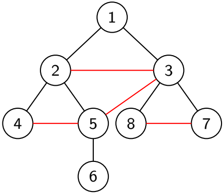


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

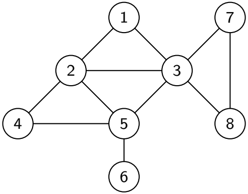


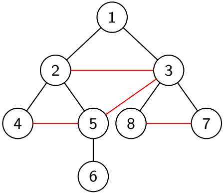


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

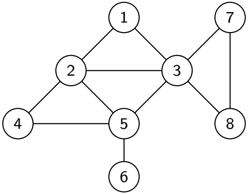


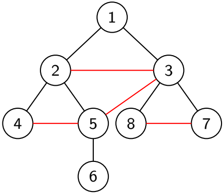


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

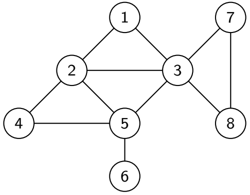


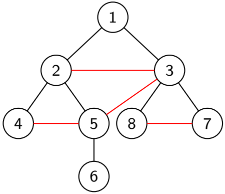


1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs

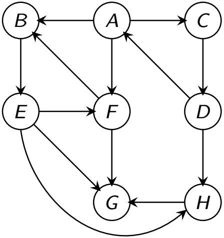




1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.

BFS: An Example in Directed Graphs



BFS with Distance

BFS(s)
    Mark all vertices as unvisited and for each v set $\mathrm{dist}(v) =$
    Initialize search tree T to be empty
    Mark vertex s as visited and set $\mathrm{dist}(s) = 0$
    set Q to be the empty queue
    enq(s)
    while Q is nonempty **do**
        $u =$ **deq**(Q)
        for each vertex $v \in \mathrm{Adj}(u)$ **do**
            if v is not visited **do**
                add edge (u, v) to T
                Mark v as visited, **enq**(v)
                and set $\mathrm{dist}(v) = \mathrm{dist}(u) + 1$

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**(s)

1. $V($ **BFS** *tree comp.*$)$ = *set vertices in connected component* s.
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v.
3. $\forall u \in V$, $\mathrm{dist}(u)$ = *the length of shortest path from* s *to* u.
4. *If* $u, v \in$ *connected component of* s, *and* $e = uv$ *is an edge of* G, *then either* $e \in$ **BFS** *tree, or* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.

Proof.

Exercise. □

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**(s)

1. $V($ **BFS** *tree comp.*$)$ = *set vertices in connected component* s.
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v.
3. $\forall u \in V$, $\mathrm{dist}(u) =$ *the length of shortest path from* s *to* u.
4. *If* $u, v \in$ *connected component of* s, *and* $e = uv$ *is an edge of* G, *then either* $e \in$ **BFS** *tree, or* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.

Proof.

Exercise. □

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**(s)

1. $V($ **BFS** *tree comp.*$)$ = *set vertices in connected component* s.
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v.
3. $\forall u \in V$, $\mathrm{dist}(u)$ = *the length of shortest path from* s *to* u.
4. *If* $u, v \in$ *connected component of* s, *and* $e = uv$ *is an edge of* G, *then either* $e \in$ **BFS** *tree, or* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.

Proof.

Exercise. □

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**(s)

1. $V($ **BFS** *tree comp.*$)$ = *set vertices in connected component* s.
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v.
3. $\forall u \in V$, $\mathrm{dist}(u)$ = *the length of shortest path from* s *to* u.
4. *If* $u, v \in$ *connected component of* s, *and* $e = uv$ *is an edge of* G, *then either* $e \in$ **BFS** *tree, or* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.

Proof.

Exercise. □

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**(s)

1. $V($ **BFS** *tree comp.*$)$ = *set vertices in connected component* s.
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v.
3. $\forall u \in V$, $\mathrm{dist}(u) =$ *the length of shortest path from* s *to* u.
4. *If* $u, v \in$ *connected component of* s, *and* $e = uv$ *is an edge of* G, *then either* $e \in$ **BFS** *tree, or* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.

Proof.

Exercise. □

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of $T \leftarrow \mathbf{BFS}(s)$:

1. *For search tree* T. $V(T)$ = *set of vertices reachable from* s
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v
3. $\forall u \in V(T)$: $\mathrm{dist}(u)$ = *length of shortest path from* s *to* u
4. *If* u *is reachable from* s, $e = (u \to v) \in E(G)$. *Then either (i)* e *is an edge in the search tree, or (ii)* $\mathrm{dist}(v) - \mathrm{dist}(u) \leq 1$.
   Not necessarily the case that $\mathrm{dist}(u) - \mathrm{dist}(v) \leq 1$.

Proof.

Exercise. □

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of $T \leftarrow \textbf{BFS}(s)$:

1. *For search tree* T. $V(T) =$ *set of vertices reachable from* s
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v
3. $\forall u \in V(T)$: $\mathrm{dist}(u) =$ *length of shortest path from* s *to* u
4. *If* u *is reachable from* s, $e = (u \to v) \in E(G)$. *Then either (i)* e *is an edge in the search tree, or (ii)* $\mathrm{dist}(v) - \mathrm{dist}(u) \leq 1$. *Not necessarily the case that* $\mathrm{dist}(u) - \mathrm{dist}(v) \leq 1$.

Proof.

Exercise. □

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of $T \leftarrow \textbf{BFS}(s)$:

1. *For search tree* T. $V(T) =$ *set of vertices reachable from* s
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v
3. $\forall u \in V(T)$: $\mathrm{dist}(u) =$ *length of shortest path from* s *to* u
4. *If* u *is reachable from* s, $e = (u \to v) \in E(G)$. *Then either (i)* e *is an edge in the search tree, or (ii)* $\mathrm{dist}(v) - \mathrm{dist}(u) \leq 1$. *Not necessarily the case that* $\mathrm{dist}(u) - \mathrm{dist}(v) \leq 1$.

Proof.

Exercise. □

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of $T \leftarrow \mathbf{BFS}(s)$:

1. *For search tree* T. $V(T)$ = *set of vertices reachable from* s
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v
3. $\forall u \in V(T)$: $\mathrm{dist}(u)$ = *length of shortest path from* s *to* u
4. *If* u *is reachable from* s, $e = (u \to v) \in E(G)$. *Then either (i)* e *is an edge in the search tree, or (ii)* $\mathrm{dist}(v) - \mathrm{dist}(u) \leq 1$. *Not necessarily the case that* $\mathrm{dist}(u) - \mathrm{dist}(v) \leq 1$.

Proof.

Exercise. □

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of $T \leftarrow \textbf{BFS}(s)$*:*

1. *For search tree* T. $V(T)$ = *set of vertices reachable from* s
2. *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v
3. $\forall u \in V(T)$: $\mathrm{dist}(u)$ = *length of shortest path from* s *to* u
4. *If* u *is reachable from* s, $e = (u \rightarrow v) \in E(G)$.
   Then either (i) e *is an edge in the search tree,*
   or (ii) $\mathrm{dist}(v) - \mathrm{dist}(u) \leq 1$.
   Not necessarily the case that $\mathrm{dist}(u) - \mathrm{dist}(v) \leq 1$.

Proof.

Exercise. □

BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
            initialize L_{i+1} to be an empty list
            for each u in L_i do
                for each edge (u, v) ∈ Adj(u) do
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
```

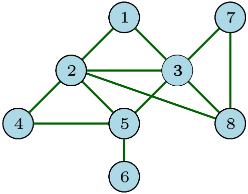
Running time: $O(n + m)$

BFS with Layers

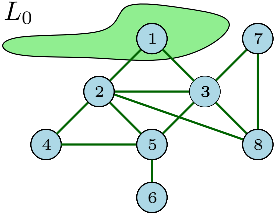
```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
            initialize L_{i+1} to be an empty list
            for each u in L_i do
                for each edge (u, v) ∈ Adj(u) do
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
```

Running time: $O(n + m)$

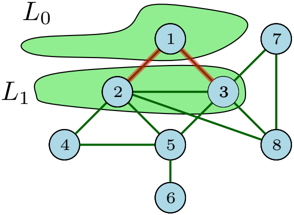
Example



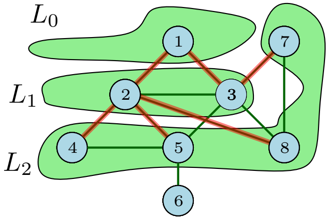
Example



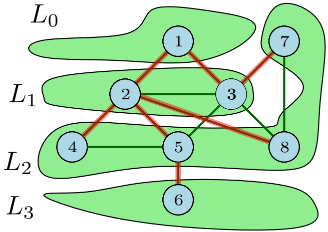
Example



Example



Example



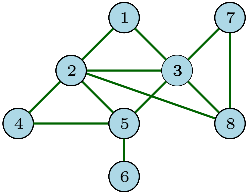
BFS with Layers: Properties

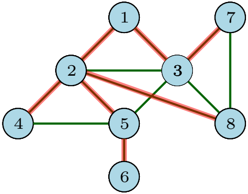
Proposition

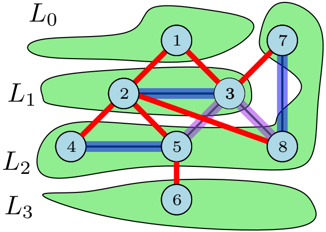
The following properties hold on termination of **BFSLayers**(s).

1. **BFSLayers**(s) *outputs a* **BFS** *tree*
2. L_i *is the set of vertices at distance exactly* i *from* s
3. *If* G *is undirected, each edge* $e = uv$ *is one of three types:*
   1. **tree** *edge between two* consecutive *layers*
   2. *non-tree* **forward**/**backward** *edge between two consecutive layers*
   3. *non-tree* **cross-edge** *with both* u, v *in same layer*
   4. $\implies$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

Example: Tree/cross/forward (backward) edges







BFS with Layers: Properties

For directed graphs

Proposition

The following properties hold on termination of **BFSLayers**(s)*, if* G *is directed.*

For each edge $e = (u \to v)$ *is one of four types:*

1. *a* **tree** *edge between consecutive layers,* $u \in L_i, v \in L_{i+1}$ *for some* $i \geq 0$
2. *a non-tree* **forward** *edge between consecutive layers*
3. *a non-tree* **backward** *edge*
4. *a* **cross-edge** *with both* u, v *in same layer*

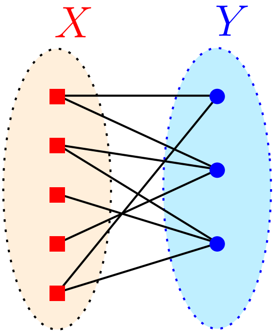
Part II

Bipartite Graphs and an application of BFS

Bipartite Graphs

Definition (Bipartite Graph)

Undirected graph $G = (V, E)$ is a **bipartite graph** if V can be partitioned into X and Y s.t. all edges in E are between X and Y.



Bipartite Graph Characterization

Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree T at some node r. Let L_i be all nodes at level i, that is, L_i is all nodes at distance i from root r. Now define X to be all nodes at even levels and Y to be all nodes at odd level. Only edges in T are between levels. □

Proposition

An odd length cycle is not bipartite.

Bipartite Graph Characterization

Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree T at some node r. Let L_i be all nodes at level i, that is, L_i is all nodes at distance i from root r. Now define X to be all nodes at even levels and Y to be all nodes at odd level. Only edges in T are between levels. □

Proposition

An odd length cycle is not bipartite.

Bipartite Graph Characterization

Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree T at some node r. Let L_i be all nodes at level i, that is, L_i is all nodes at distance i from root r. Now define X to be all nodes at even levels and Y to be all nodes at odd level. Only edges in T are between levels. □

Proposition

An odd length cycle is not bipartite.

Bipartite Graph Characterization

Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree T at some node r. Let L_i be all nodes at level i, that is, L_i is all nodes at distance i from root r. Now define X to be all nodes at even levels and Y to be all nodes at odd level. Only edges in T are between levels. □

Proposition

An odd length cycle is not bipartite.

Odd Cycles are not Bipartite

Proposition

An odd length cycle is not bipartite.

Proof.

Let $C = u_1, u_2, \ldots, u_{2k+1}, u_1$ be an odd cycle. Suppose C is a bipartite graph and let X, Y be the partition. Without loss of generality $u_1 \in X$. Implies $u_2 \in Y$. Implies $u_3 \in X$. Inductively, $u_i \in X$ if i is odd $u_i \in Y$ if i is even. But $\{u_1, u_{2k+1}\}$ is an edge and both belong to X! □

Subgraphs

Definition

Given a graph $G = (V, E)$ a **subgraph** of G is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

Proposition

If an undirected G is bipartite then any subgraph H of G is also bipartite.

Proposition

An undirected graph G is not bipartite if G has an odd cycle C as a subgraph.

Proof.

If G is bipartite then since C is a subgraph, C is also bipartite (by

Subgraphs

Definition

Given a graph $G = (V, E)$ a **subgraph** of G is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

Proposition

If an undirected G is bipartite then any subgraph H of G is also bipartite.

Proposition

An undirected graph G is not bipartite if G has an odd cycle C as a subgraph.

Proof.

If G is bipartite then since C is a subgraph, C is also bipartite (by

Subgraphs

Definition

Given a graph $G = (V, E)$ a **subgraph** of G is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

Proposition

If an undirected G is bipartite then any subgraph H of G is also bipartite.

Proposition

An undirected graph G is not bipartite if G has an odd cycle C as a subgraph.

Proof.

If G is bipartite then since C is a subgraph, C is also bipartite (by

Subgraphs

Definition

Given a graph $G = (V, E)$ a **subgraph** of G is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

Proposition

If an undirected G is bipartite then any subgraph H of G is also bipartite.

Proposition

An undirected graph G is not bipartite if G has an odd cycle C as a subgraph.

Proof.

If G is bipartite then since C is a subgraph, C is also bipartite (by

Bipartite Graph Characterization

Theorem

An undirected graph G *is bipartite* $\iff$ *it has no odd length cycle as subgraph.*

Proof.

Only If: G has an odd cycle implies G is not bipartite.

If: G has no odd length cycle. Assume without loss of generality that G is connected.

1. Pick u arbitrarily and do **BFS**(u)
2. $X = \cup_{i \text{ is even}} L_i$ and $Y = \cup_{i \text{ is odd}} L_i$
3. **Claim:** X and Y is a valid partition if G has no odd length cycle.

□

Bipartite Graph Characterization

Theorem

An undirected graph G is bipartite $\iff$ it has no odd length cycle as subgraph.

Proof.

Only If: G has an odd cycle implies G is not bipartite.
If: G has no odd length cycle. Assume without loss of generality that G is connected.

1. Pick u arbitrarily and do **BFS**(u)
2. $X = \cup_{i \text{ is even}} L_i$ and $Y = \cup_{i \text{ is odd}} L_i$
3. **Claim:** X and Y is a valid partition if G has no odd length cycle.

□

Proof of Claim

Claim

In **BFS**(u) *if* $a, b \in L_i$ *and* $ab \in E(G)$ *then there is an odd length cycle containing* ab.

Proof.

Let v be least common ancestor of a, b in **BFS** tree T.
v is in some level $j < i$ (could be u itself).
Path from $v \rightsquigarrow a$ in T is of length $j - i$.
Path from $v \rightsquigarrow b$ in T is of length $j - i$.
These two paths plus (a, b) forms an odd cycle of length $2(j - i) + 1$. □

Proof of Claim

Claim

In **BFS**(u) *if* $a, b \in L_i$ *and* $ab \in E(G)$ *then there is an odd length cycle containing* ab*.*

Proof.

Let v be least common ancestor of a, b in **BFS** tree T.
v is in some level $j < i$ (could be u itself).
Path from $v \rightsquigarrow a$ in T is of length $j - i$.
Path from $v \rightsquigarrow b$ in T is of length $j - i$.
These two paths plus (a, b) forms an odd cycle of length $2(j - i) + 1$. □

Proof of Claim: Figure

Another tidbit

Corollary

There is an $O(n+m)$ *time algorithm to check if* G *is bipartite and output an odd cycle if it is not.*

Part III

Shortest Paths and Dijkstra's Algorithm

Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u \to v)$, $\ell(e) = \ell(u \to v)$ is its length.

1. Given nodes s, t find shortest path from s to t.
2. Given node s find shortest path from s to all other nodes.
3. Find shortest paths for all pairs of nodes.

Many applications!

Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u \to v)$, $\ell(e) = \ell(u \to v)$ is its length.

1. Given nodes s, t find shortest path from s to t.
2. Given node s find shortest path from s to all other nodes.
3. Find shortest paths for all pairs of nodes.

Many applications!

Single-Source Shortest Paths:

Non-Negative Edge Lengths

Single-Source Shortest Path Problems

1. Input: A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u \to v)$, $\ell(e) = \ell(u \to v)$ is its length.
2. Given nodes s, t find shortest path from s to t.
3. Given node s find shortest path from s to all other nodes.

1. Restrict attention to directed graphs
2. Undirected graph problem can be reduced to directed graph problem - how?
   1. Given undirected graph G, create a new directed graph G' by replacing each edge $\{u, v\}$ in G by $(u \to v)$ and (v, u) in G'.
   2. set $\ell(u \to v) = \ell(v, u) = \ell(\{u, v\})$
   3. Exercise: show reduction works

Single-Source Shortest Path Problems

1. Input: A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u \to v)$, $\ell(e) = \ell(u \to v)$ is its length.
2. Given nodes s, t find shortest path from s to t.
3. Given node s find shortest path from s to all other nodes.

1. Restrict attention to directed graphs
2. Undirected graph problem can be reduced to directed graph problem - how?
   1. Given undirected graph G, create a new directed graph G' by replacing each edge $\{u, v\}$ in G by $(u \to v)$ and (v, u) in G'.
   2. set $\ell(u \to v) = \ell(v, u) = \ell(\{u, v\})$
   3. Exercise: show reduction works

Single-Source Shortest Path Problems

1. Input: A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u \to v)$, $\ell(e) = \ell(u \to v)$ is its length.
2. Given nodes s, t find shortest path from s to t.
3. Given node s find shortest path from s to all other nodes.

1. Restrict attention to directed graphs
2. Undirected graph problem can be reduced to directed graph problem - how?
   1. Given undirected graph G, create a new directed graph G' by replacing each edge $\{u, v\}$ in G by $(u \to v)$ and (v, u) in G'.
   2. set $\ell(u \to v) = \ell(v, u) = \ell(\{u, v\})$
   3. Exercise: show reduction works

Single-Source Shortest Paths via BFS

1. **Special case:** All edge lengths are 1.
   1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
   2. $O(m + n)$ time algorithm.
2. **Special case:** Suppose $\ell(e)$ is an integer for all e?
   Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e.
3. Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

Single-Source Shortest Paths via BFS

1. **Special case:** All edge lengths are 1.
   1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
   2. $O(m+n)$ time algorithm.
2. **Special case:** Suppose $\ell(e)$ is an integer for all e? Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e.
3. Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL+n)$ nodes. **BFS** takes $O(mL+n)$ time. Not efficient if L is large.

Single-Source Shortest Paths via BFS

1. **Special case:** All edge lengths are 1.
   1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
   2. $O(m+n)$ time algorithm.
2. **Special case:** Suppose $\ell(e)$ is an integer for all e?
   Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e.
3. Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

Single-Source Shortest Paths via BFS

1. **Special case:** All edge lengths are 1.
   1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
   2. $O(m + n)$ time algorithm.
2. **Special case:** Suppose $\ell(e)$ is an integer for all e?
   Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e.
3. Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

Single-Source Shortest Paths via BFS

1. **Special case:** All edge lengths are $\mathbf{1}$.
   1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
   2. $O(m+n)$ time algorithm.
2. **Special case:** Suppose $\ell(e)$ is an integer for all e? Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e.
3. Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL+n)$ nodes. **BFS** takes $O(mL+n)$ time. Not efficient if L is large.

Single-Source Shortest Paths via BFS

1. **Special case:** All edge lengths are $\mathbf{1}$.
   1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
   2. $O(m + n)$ time algorithm.
2. **Special case:** Suppose $\ell(e)$ is an integer for all e? Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e.
3. Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

Towards an algorithm

Why does **BFS** work?

BFS(s) explores nodes in increasing distance from s

Lemma

Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

1. $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ *is a shortest path from s to v_i*
2. $\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$.

Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$. Then

Towards an algorithm

Why does **BFS** work?
BFS(s) explores nodes in increasing distance from s

Lemma

Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

1. *$s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from s to v_i*
2. *$\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$.*

Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$. Then

Towards an algorithm

Why does **BFS** work?

BFS(s) explores nodes in increasing distance from s

Lemma

Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is a shortest path from s to v_k then for $1 \le i < k$:

1. *$s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ is a shortest path from s to v_i*
2. *$\mathrm{dist}(s, v_i) \le \mathrm{dist}(s, v_k)$.*

Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then

Towards an algorithm

Lemma

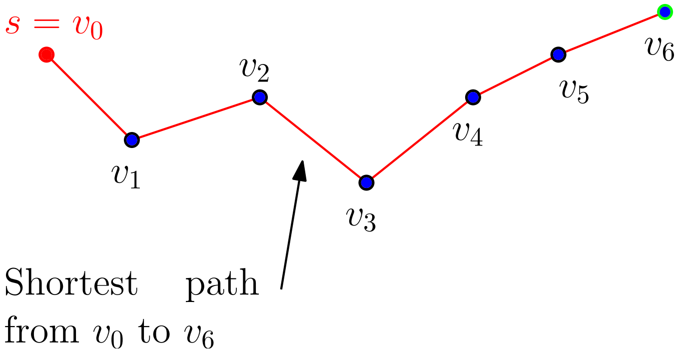
Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

1. $s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ *is a shortest path from* s *to* v_i
2. $\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$.

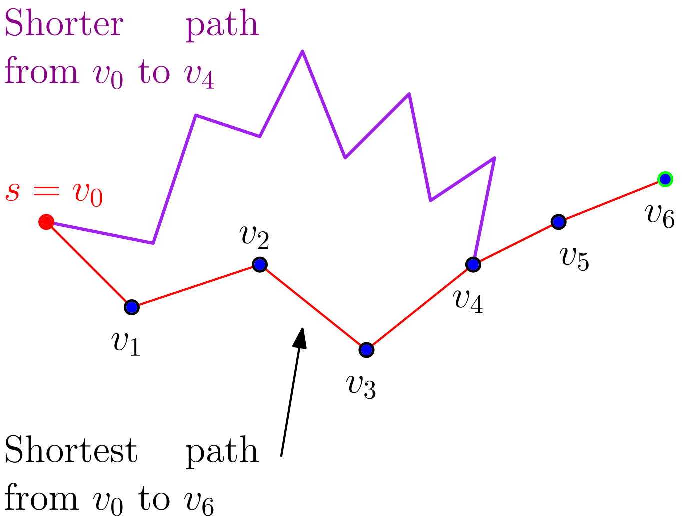
Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then P' concatenated with $v_i \to v_{i+1} \ldots \to v_k$ contains a strictly shorter path to v_k than $s = v_0 \to v_1 \ldots \to v_k$. □

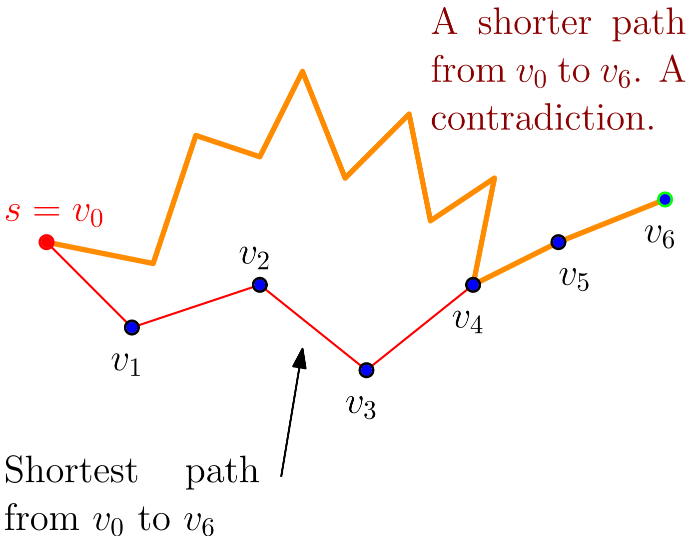
A proof by picture



A proof by picture



A proof by picture



A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

```
Initialize for each node v, dist(s, v) = ∞
Initialize S = ∅,
for i = 1 to |V| do
    (* Invariant:  S contains the i − 1 closest nodes to s *)
    Among nodes in V \ S, find the node v that is the
            ith closest to s
    Update dist(s, v)
    S = S ∪ {v}
```

How can we implement the step in the for loop?

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

```
Initialize for each node v, dist(s, v) = ∞
Initialize S = ∅,
for i = 1 to |V| do
    (* Invariant: S contains the i − 1 closest nodes to s *)
    Among nodes in V \ S, find the node v that is the
          ith closest to s
    Update dist(s, v)
    S = S ∪ {v}
```

How can we implement the step in the for loop?

Finding the ith closest node

1. S contains the $i - 1$ closest nodes to s
2. Want to find the ith closest node from $V - S$.

What do we know about the ith closest node?

Claim

Let P be a shortest path from s to v where v is the ith closest node. Then, all intermediate nodes in P belong to S.

Proof.

If P had an intermediate node u not in S then u will be closer to s than v. Implies v is not the ith closest node to s - recall that S already has the $i - 1$ closest nodes. □

Finding the ith closest node

1. S contains the $i - 1$ closest nodes to s
2. Want to find the ith closest node from $V - S$.

What do we know about the ith closest node?

Claim

Let P be a shortest path from s to v where v is the ith closest node. Then, all intermediate nodes in P belong to S.

Proof.

If P had an intermediate node u not in S then u will be closer to s than v. Implies v is not the ith closest node to s - recall that S already has the $i - 1$ closest nodes. □

Finding the ith closest node

1. S contains the $i - 1$ closest nodes to s
2. Want to find the ith closest node from $V - S$.

What do we know about the ith closest node?

Claim

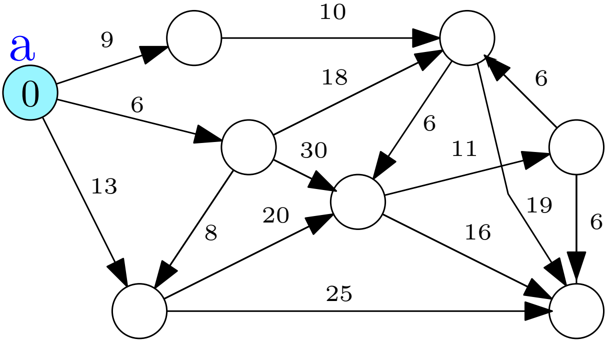
Let P be a shortest path from s to v where v is the ith closest node. Then, all intermediate nodes in P belong to S.

Proof.

If P had an intermediate node u not in S then u will be closer to s than v. Implies v is not the ith closest node to s - recall that S already has the $i - 1$ closest nodes. □

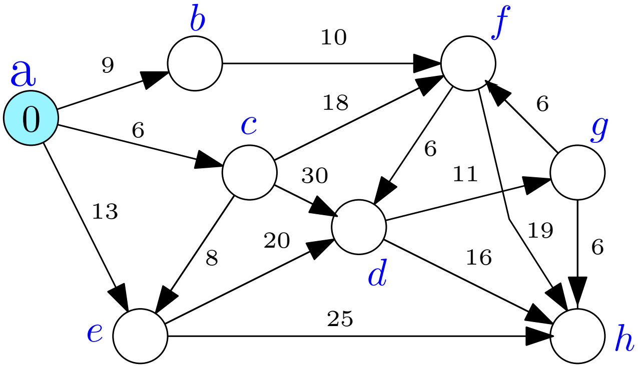
Finding the ith closest node repeatedly

An example



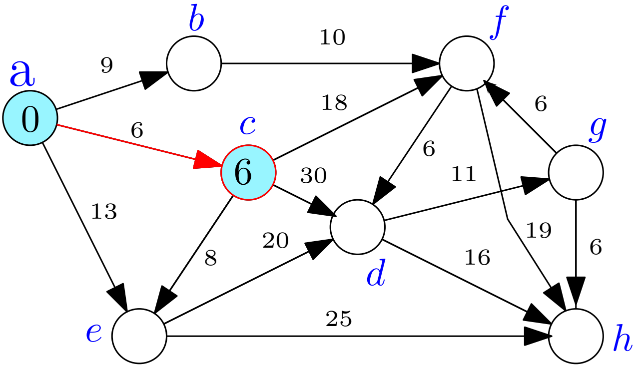
Finding the ith closest node repeatedly

An example



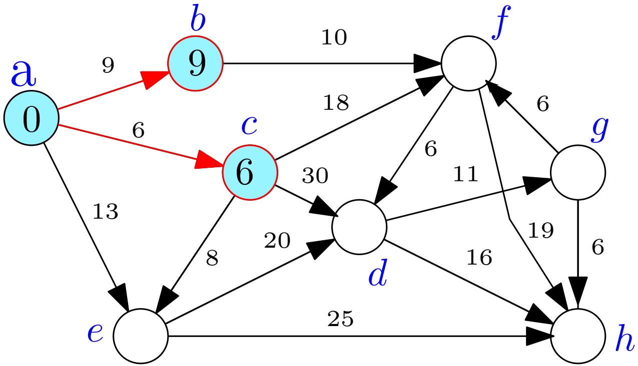
Finding the ith closest node repeatedly

An example



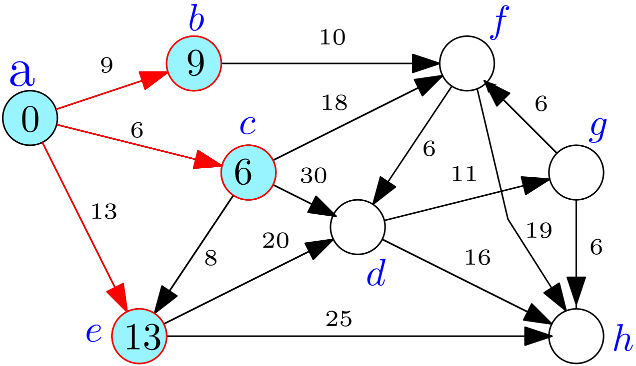
Finding the ith closest node repeatedly

An example



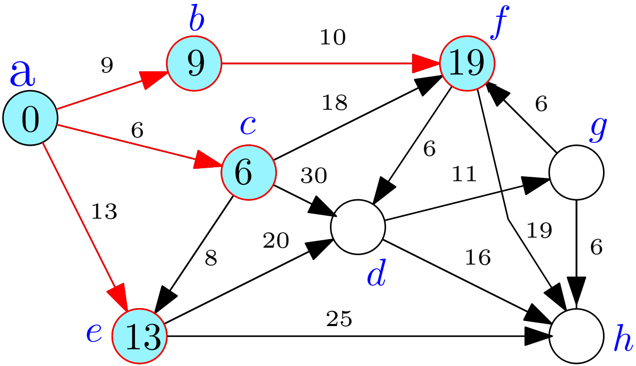
Finding the ith closest node repeatedly

An example



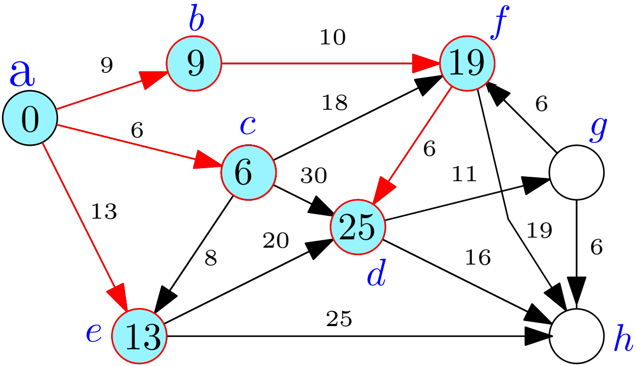
Finding the ith closest node repeatedly

An example



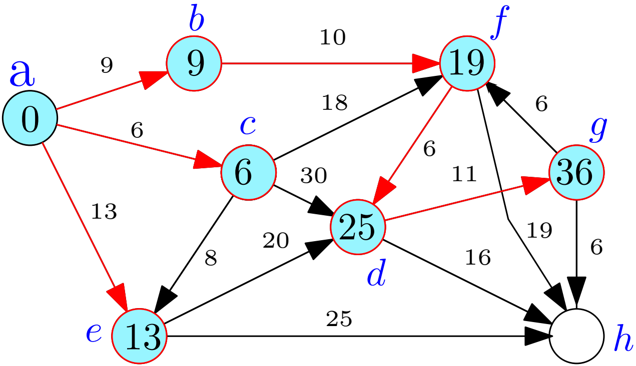
Finding the ith closest node repeatedly

An example



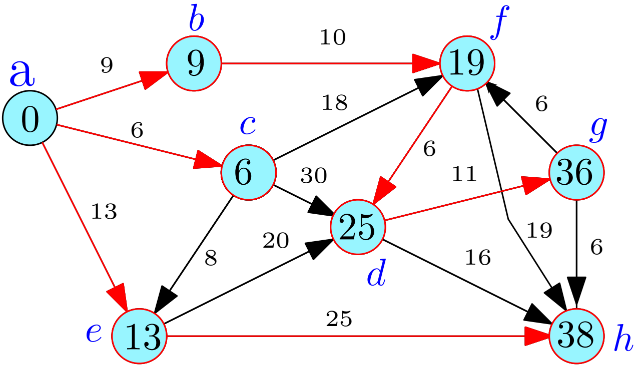
Finding the ith closest node repeatedly

An example

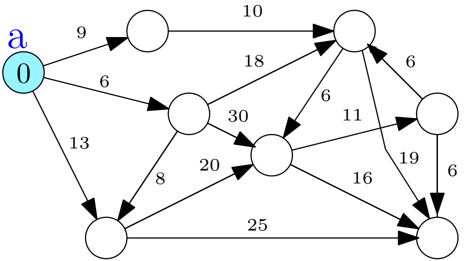


Finding the ith closest node repeatedly

An example



Finding the ith closest node



Corollary

The ith closest node is adjacent to S.

Finding the ith closest node

1. S contains the $i-1$ closest nodes to s
2. Want to find the ith closest node from $V - S$.
3. For each $u \in V \setminus S$ let $P(s, u, S)$ be a shortest path from s to u using only nodes in S as intermediate vertices.
4. Let $d'(s, u)$ be the length of $P(s, u, S)$
5. Observations: for each $u \in V - S$,
    1. $\mathrm{dist}(s, u) \leq d'(s, u)$ since we are constraining the paths
    2. $d'(s, u) = \min_{a \in S}(\mathrm{dist}(s, a) + \ell(a, u))$ - Why?
6. **Lemma**

   If v is the ith closest node to s, then $d'(s, v) = \mathrm{dist}(s, v)$.

Finding the ith closest node

1. S contains the $i - 1$ closest nodes to s
2. Want to find the ith closest node from $V - S$.
3. For each $u \in V \setminus S$ let $P(s, u, S)$ be a shortest path from s to u using only nodes in S as intermediate vertices.
4. Let $d'(s, u)$ be the length of $P(s, u, S)$
5. Observations: for each $u \in V - S$,
    1. $\mathrm{dist}(s, u) \leq d'(s, u)$ since we are constraining the paths
    2. $d'(s, u) = \min_{a \in S}(\mathrm{dist}(s, a) + \ell(a, u))$ - Why?
6. **Lemma**

   If v is the ith closest node to s, then $d'(s, v) = \mathrm{dist}(s, v)$.

Finding the ith closest node

1. S contains the $i - 1$ closest nodes to s
2. Want to find the ith closest node from $V - S$.
3. For each $u \in V \setminus S$ let $P(s, u, S)$ be a shortest path from s to u using only nodes in S as intermediate vertices.
4. Let $d'(s, u)$ be the length of $P(s, u, S)$
5. Observations: for each $u \in V - S$,
    1. $\mathrm{dist}(s, u) \le d'(s, u)$ since we are constraining the paths
    2. $d'(s, u) = \min_{a \in S}(\mathrm{dist}(s, a) + \ell(a, u))$ - Why?
6. **Lemma**

    If v is the ith closest node to s, then $d'(s, v) = \mathrm{dist}(s, v)$.

Finding the ith closest node

1. S contains the $i-1$ closest nodes to s
2. Want to find the ith closest node from $V-S$.
3. For each $u \in V \setminus S$ let $P(s,u,S)$ be a shortest path from s to u using only nodes in S as intermediate vertices.
4. Let $d'(s,u)$ be the length of $P(s,u,S)$
5. Observations: for each $u \in V-S$,
   1. $\mathrm{dist}(s,u) \leq d'(s,u)$ since we are constraining the paths
   2. $d'(s,u) = \min_{a \in S}(\mathrm{dist}(s,a) + \ell(a,u))$ - Why?
6. **Lemma**

   If v is the ith closest node to s, then $d'(s,v) = \mathrm{dist}(s,v)$.

Finding the ith closest node

1. S contains the $i-1$ closest nodes to s
2. Want to find the ith closest node from $V-S$.
3. For each $u \in \mathrm{V} \setminus S$ let $P(s,u,S)$ be a shortest path from s to u using only nodes in S as intermediate vertices.
4. Let $d'(s,u)$ be the length of $P(s,u,S)$
5. Observations: for each $u \in V-S$,
   1. $\mathrm{dist}(s,u) \le d'(s,u)$ since we are constraining the paths
   2. $d'(s,u) = \min_{a \in S}(\mathrm{dist}(s,a) + \ell(a,u))$ - Why?
6. **Lemma**

   If v is the ith closest node to s, then $d'(s,v) = \mathrm{dist}(s,v)$.

Finding the ith closest node

1. S contains the $i-1$ closest nodes to s
2. Want to find the ith closest node from $V-S$.
3. For each $u \in \mathrm{V} \setminus S$ let $P(s,u,S)$ be a shortest path from s to u using only nodes in S as intermediate vertices.
4. Let $d'(s,u)$ be the length of $P(s,u,S)$
5. Observations: for each $u \in V-S$,
   1. $\mathrm{dist}(s,u) \leq d'(s,u)$ since we are constraining the paths
   2. $d'(s,u) = \min_{a \in S}(\mathrm{dist}(s,a) + \ell(a,u))$ - Why?
6. **Lemma**

   If v is the ith closest node to s, then $d'(s,v) = \mathrm{dist}(s,v)$.

Finding the ith closest node

Lemma

Given:

1. S: *Set of* $i-1$ *closest nodes to* s.
2. $d'(s,u) = \min_{x \in S}(\mathrm{dist}(s,x) + \ell(x,u))$

If v *is an* i*th closest node to* s*, then* $d'(s,v) = \mathrm{dist}(s,v)$.

Proof.

Let v be the ith closest node to s. Then there is a shortest path P from s to v that contains only nodes in S as intermediate nodes (see previous claim). Therefore $d'(s,v) = \mathrm{dist}(s,v)$. □

Finding the ith closest node

Lemma

If v is an ith closest node to s, then $d'(s, v) = \mathrm{dist}(s, v)$.

Corollary

The ith closest node to s is the node $v \in V - S$ such that $d'(s, v) = \min_{u \in V - S} d'(s, u)$.

Proof.

For every node $u \in V - S$, $\mathrm{dist}(s, u) \leq d'(s, u)$ and for the ith closest node v, $\mathrm{dist}(s, v) = d'(s, v)$. Moreover, $\mathrm{dist}(s, u) \geq \mathrm{dist}(s, v)$ for each $u \in V - S$. □

Candidate algorithm for shortest path

```
Initialize for each node v:  dist(s, v) = ∞
Initialize S = ∅, d'(s, s) = 0
for i = 1 to |V| do
    (* Invariant: S contains the i-1 closest nodes to s *)
    (* Invariant: d'(s,u) is shortest path distance from u to s
     using only S as intermediate nodes*)
    Let v be such that d'(s, v) = min_{u∈V−S} d'(s, u)
    dist(s, v) = d'(s, v)
    S = S ∪ {v}
    for each node u in V \ S do
        d'(s, u) ⇐ min_{a∈S} (dist(s, a) + ℓ(a, u))
```

Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

Candidate algorithm for shortest path

```
Initialize for each node v:  dist(s, v) = ∞
Initialize S = ∅, d'(s, s) = 0
for i = 1 to |V| do
    (* Invariant:  S contains the i-1 closest nodes to s *)
    (* Invariant:  d'(s,u) is shortest path distance from u to s
     using only S as intermediate nodes*)
    Let v be such that d'(s, v) = min_{u∈V−S} d'(s, u)
    dist(s, v) = d'(s, v)
    S = S ∪ {v}
    for each node u in V \ S do
        d'(s, u) ⇐ min_{a∈S} (dist(s, a) + ℓ(a, u))
```

Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

Candidate algorithm for shortest path

```
Initialize for each node v:  dist(s, v) = ∞
Initialize S = ∅, d'(s, s) = 0
for i = 1 to |V| do
    (* Invariant:  S contains the i-1 closest nodes to s *)
    (* Invariant:  d'(s,u) is shortest path distance from u to s
     using only S as intermediate nodes*)
    Let v be such that d'(s, v) = min_{u ∈ V−S} d'(s, u)
    dist(s, v) = d'(s, v)
    S = S ∪ {v}
    for each node u in V \ S do
        d'(s, u) ⇐ min_{a ∈ S} (dist(s, a) + ℓ(a, u))
```

Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

Candidate algorithm for shortest path

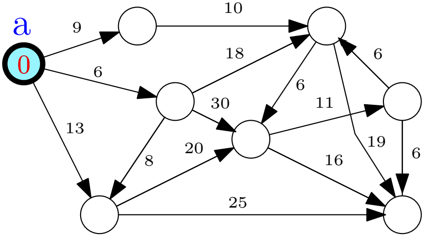
```
Initialize for each node v:  dist(s, v) = ∞
Initialize S = ∅,  d'(s, s) = 0
for i = 1 to |V| do
    (* Invariant:  S contains the i-1 closest nodes to s *)
    (* Invariant:  d'(s,u) is shortest path distance from u to s
     using only S as intermediate nodes*)
    Let v be such that d'(s, v) = min_{u∈V−S} d'(s, u)
    dist(s, v) = d'(s, v)
    S = S ∪ {v}
    for each node u in V \ S do
        d'(s, u) ⇐ min_{a∈S} (dist(s, a) + ℓ(a, u))
```

Correctness: By induction on i using previous lemmas.

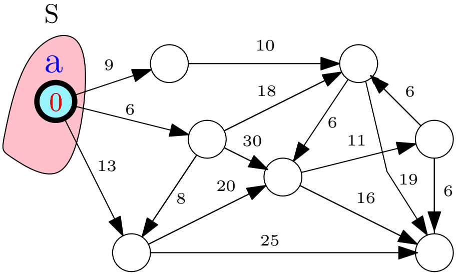
Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

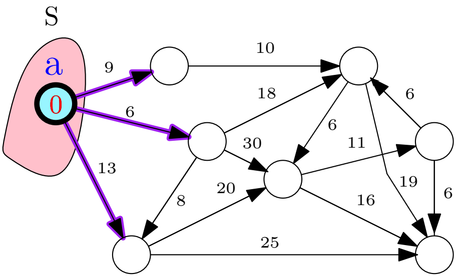
Example



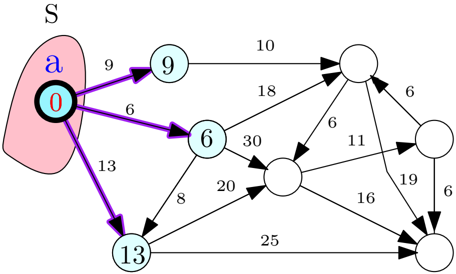
Example



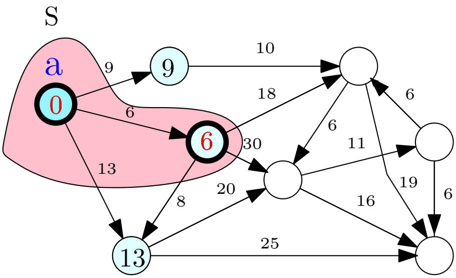
Example



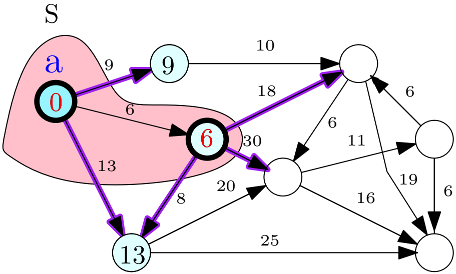
Example



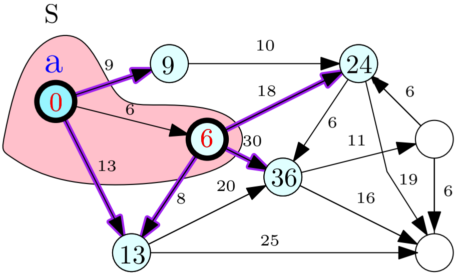
Example



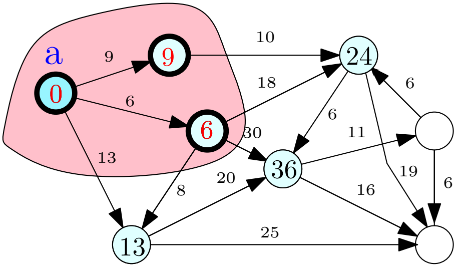
Example



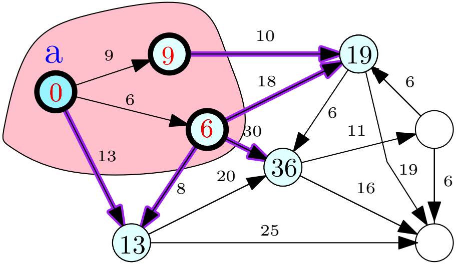
Example



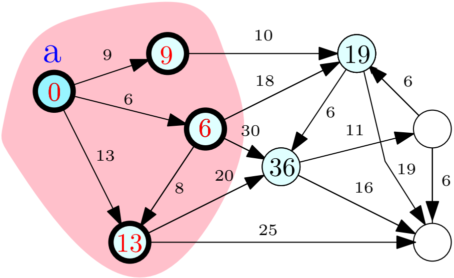
Example



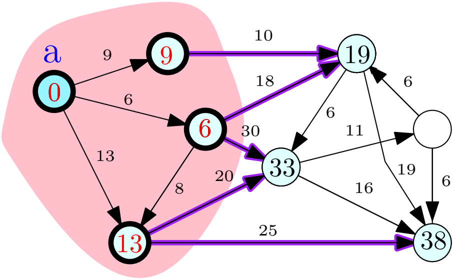
Example



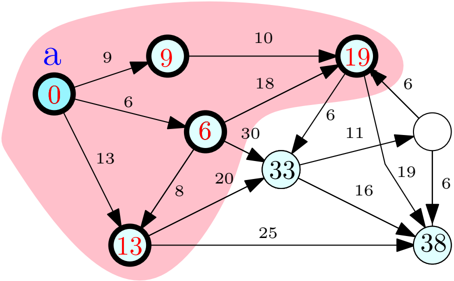
Example



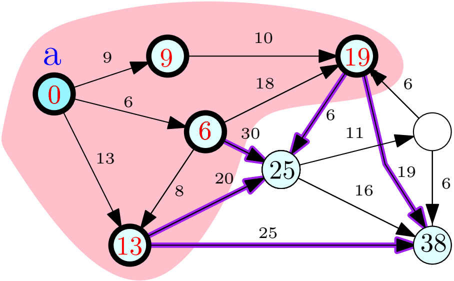
Example



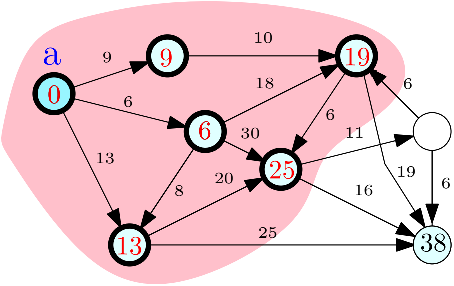
Example



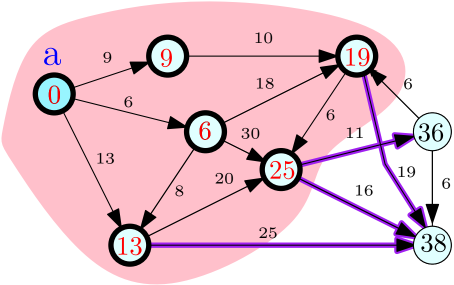
Example



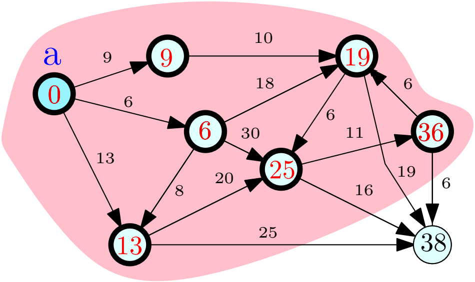
Example



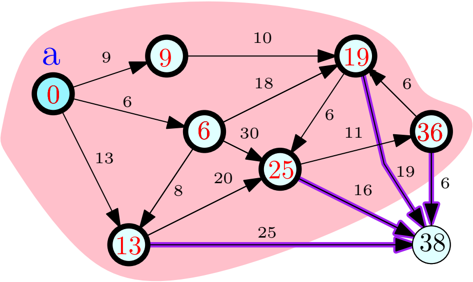
Example



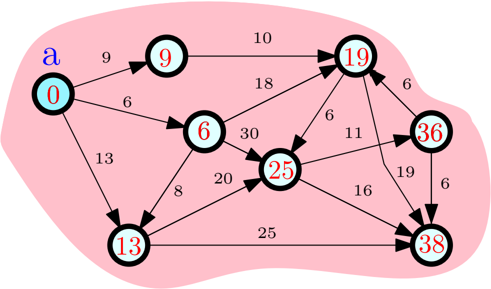
Example



Example



Example



Improved Algorithm

1. Main work is to compute the $d'(s,u)$ values in each iteration
2. $d'(s,u)$ changes from iteration i to $i+1$ only because of the node v that is added to S in iteration i.

```
Initialize for each node v, dist(s,v) = d'(s,v) = ∞
Initialize S = ∅, d'(s,s) = 0
for i = 1 to |V| do
    // S contains the i − 1 closest nodes to s,
    //        and the values of d'(s,u) are current
    v be node realizing d'(s,v) = min_{u∈V−S} d'(s,u)
    dist(s,v) = d'(s,v)
    S = S ∪ {v}
    Update d'(s,u) for each u in V − S as follows:
        d'(s,u) = min(d'(s,u), dist(s,v) + ℓ(v,u))
```

Running time: $O(m+n^2)$ time.

1. n outer iterations and in each iteration following steps
2. updating $d'(s,u)$ after v added takes $O(deg(v))$ time so total

Improved Algorithm

1. Main work is to compute the $d'(s,u)$ values in each iteration
2. $d'(s,u)$ changes from iteration i to $i+1$ only because of the node v that is added to S in iteration i.

```
Initialize for each node v, dist(s,v) = d'(s,v) = ∞
Initialize S = ∅, d'(s,s) = 0
for i = 1 to |V| do
    // S contains the i − 1 closest nodes to s,
    //       and the values of d'(s,u) are current
    v be node realizing d'(s,v) = min_{u∈V−S} d'(s,u)
    dist(s,v) = d'(s,v)
    S = S ∪ {v}
    Update d'(s,u) for each u in V − S as follows:
        d'(s,u) = min(d'(s,u), dist(s,v) + ℓ(v,u))
```

Running time: $O(m+n^2)$ time.

1. n outer iterations and in each iteration following steps
2. updating $d'(s,u)$ after v added takes $O(deg(v))$ time so total

Improved Algorithm

```
Initialize for each node v, dist(s, v) = d'(s, v) = ∞
Initialize S = ∅, d'(s,s) = 0
for i = 1 to |V| do
    // S contains the i − 1 closest nodes to s,
    //       and the values of d'(s, u) are current
    v be node realizing d'(s, v) = min_{u∈V−S} d'(s, u)
    dist(s, v) = d'(s, v)
    S = S ∪ {v}
    Update d'(s, u) for each u in V − S as follows:
        d'(s, u) = min(d'(s, u), dist(s, v) + ℓ(v, u))
```

Running time: $O(m + n^2)$ time.

1. n outer iterations and in each iteration following steps
2. updating $d'(s, u)$ after v added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters S only once
3. Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

1. eliminate $d'(s, u)$ and let $\mathrm{dist}(s, u)$ maintain it
2. update *dist* values after adding v by scanning edges out of v

```
Initialize for each node v, dist(s,v) = ∞
Initialize S = {}, dist(s,s) = 0
for i = 1 to |V| do
    Let v be such that dist(s,v) = min_{u∈V−S} dist(s,u)
    S = S ∪ {v}
    for each u in Adj(v) do
        dist(s,u) = min(dist(s,u), dist(s,v) + ℓ(v,u))
```

Priority Queues to maintain *dist* values for faster running time

1. Using heaps and standard priority queues: $O((m+n)\log n)$
2. Using Fibonacci heaps: $O(m + n\log n)$.

Dijkstra's Algorithm

1. eliminate $d'(s, u)$ and let $\mathrm{dist}(s, u)$ maintain it
2. update *dist* values after adding v by scanning edges out of v

```
Initialize for each node v, dist(s, v) = ∞
Initialize S = {}, dist(s, s) = 0
for i = 1 to |V| do
    Let v be such that dist(s, v) = min_{u∈V−S} dist(s, u)
    S = S ∪ {v}
    for each u in Adj(v) do
        dist(s, u) = min(dist(s, u), dist(s, v) + ℓ(v, u))
```

Priority Queues to maintain *dist* values for faster running time

1. Using heaps and standard priority queues: $O((m+n)\log n)$
2. Using Fibonacci heaps: $O(m + n \log n)$.

Dijkstra's Algorithm

1. eliminate $d'(s,u)$ and let $\mathrm{dist}(s,u)$ maintain it
2. update *dist* values after adding v by scanning edges out of v

```
Initialize for each node v, dist(s, v) = ∞
Initialize S = {}, dist(s, s) = 0
for i = 1 to |V| do
    Let v be such that dist(s, v) = min_{u∈V−S} dist(s, u)
    S = S ∪ {v}
    for each u in Adj(v) do
        dist(s, u) = min(dist(s, u), dist(s, v) + ℓ(v, u))
```

Priority Queues to maintain *dist* values for faster running time

1. Using heaps and standard priority queues: $O((m+n)\log n)$
2. Using Fibonacci heaps: $O(m + n\log n)$.

Dijkstra's Algorithm

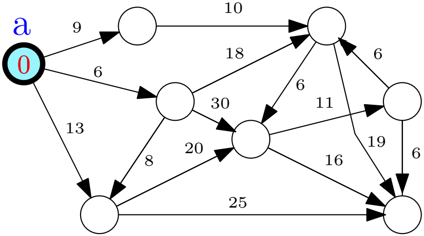
1. eliminate $d'(s, u)$ and let $\text{dist}(s, u)$ maintain it
2. update *dist* values after adding v by scanning edges out of v

```
Initialize for each node v, dist(s, v) = ∞
Initialize S = {}, dist(s, s) = 0
for i = 1 to |V| do
    Let v be such that dist(s, v) = min_{u∈V−S} dist(s, u)
    S = S ∪ {v}
    for each u in Adj(v) do
        dist(s, u) = min(dist(s, u), dist(s, v) + ℓ(v, u))
```

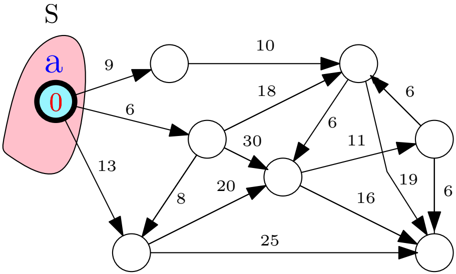
Priority Queues to maintain *dist* values for faster running time

1. Using heaps and standard priority queues: $O((m + n) \log n)$
2. Using Fibonacci heaps: $O(m + n \log n)$.

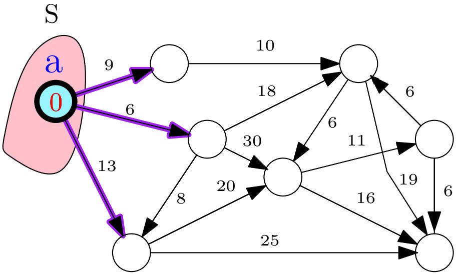
Example: Dijkstra algorithm in action



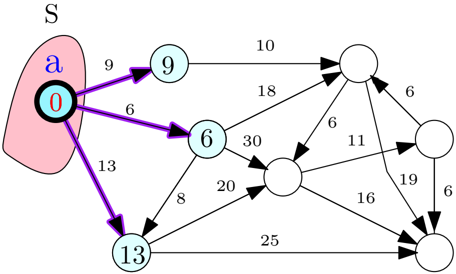
Example: Dijkstra algorithm in action



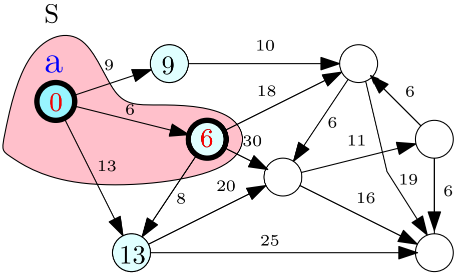
Example: Dijkstra algorithm in action



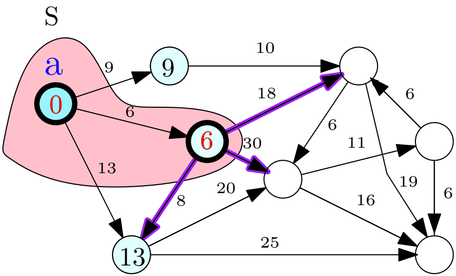
Example: Dijkstra algorithm in action



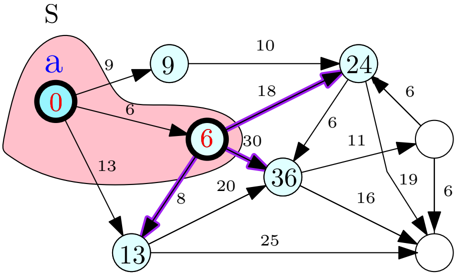
Example: Dijkstra algorithm in action



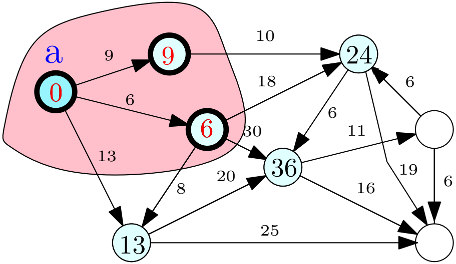
Example: Dijkstra algorithm in action



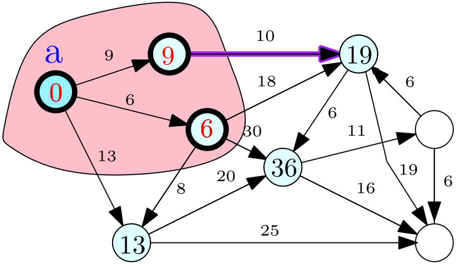
Example: Dijkstra algorithm in action



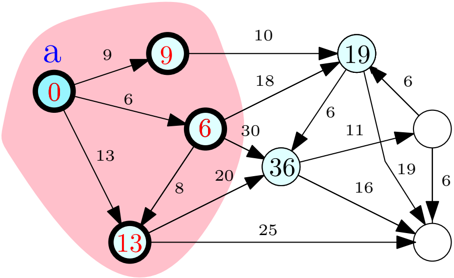
Example: Dijkstra algorithm in action



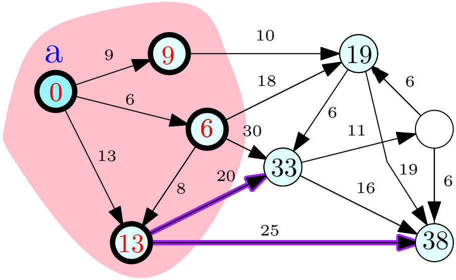
Example: Dijkstra algorithm in action



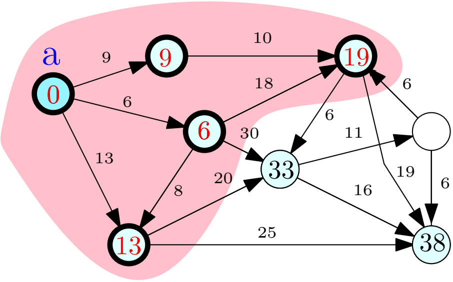
Example: Dijkstra algorithm in action



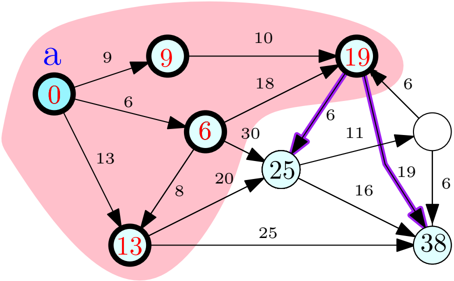
Example: Dijkstra algorithm in action



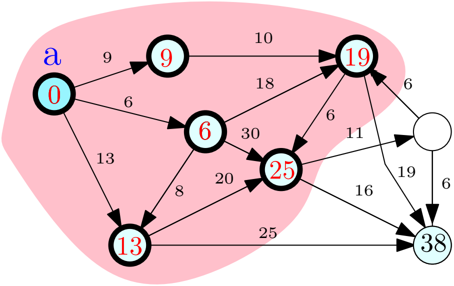
Example: Dijkstra algorithm in action



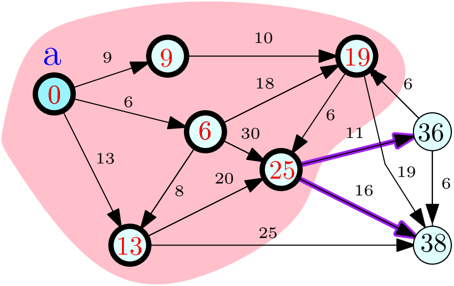
Example: Dijkstra algorithm in action



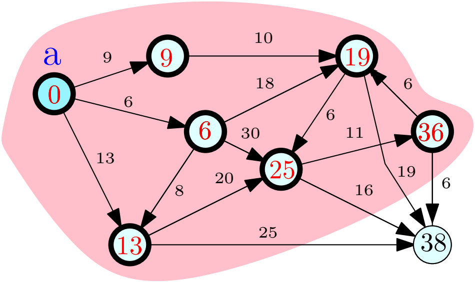
Example: Dijkstra algorithm in action



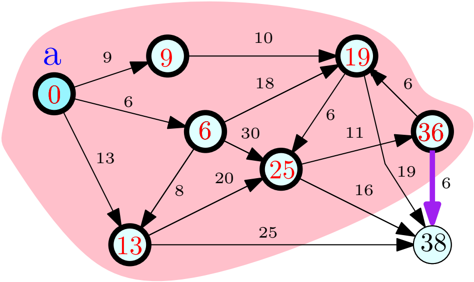
Example: Dijkstra algorithm in action



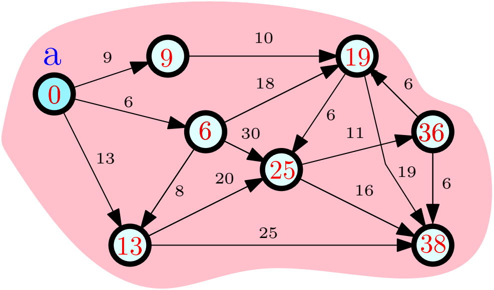
Example: Dijkstra algorithm in action



Example: Dijkstra algorithm in action



Example: Dijkstra algorithm in action



Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

1. **makePQ**: create an empty queue.
2. **findMin**: find the minimum key in S.
3. **extractMin**: Remove $v \in S$ with smallest key and return it.
4. **insert**($v, k(v)$): Add new element v with key $k(v)$ to S.
5. **delete**(v): Remove element v from S.
6. **decreaseKey**($v, k'(v)$): *decrease* key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$.
7. **meld**: merge two separate priority queues into one.

All operations can be performed in $O(\log n)$ time.
decreaseKey is implemented via **delete** and **insert**.

Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

1. **makePQ**: create an empty queue.
2. **findMin**: find the minimum key in S.
3. **extractMin**: Remove $v \in S$ with smallest key and return it.
4. **insert**$(v, k(v))$: Add new element v with key $k(v)$ to S.
5. **delete**(v): Remove element v from S.
6. **decreaseKey**$(v, k'(v))$: *decrease* key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$.
7. **meld**: merge two separate priority queues into one.

All operations can be performed in $O(\log n)$ time.
decreaseKey is implemented via **delete** and **insert**.

Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

1. **makePQ**: create an empty queue.
2. **findMin**: find the minimum key in S.
3. **extractMin**: Remove $v \in S$ with smallest key and return it.
4. **insert**$(v, k(v))$: Add new element v with key $k(v)$ to S.
5. **delete**(v): Remove element v from S.
6. **decreaseKey**$(v, k'(v))$: *decrease* key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$.
7. **meld**: merge two separate priority queues into one.

All operations can be performed in $O(\log n)$ time.

decreaseKey is implemented via **delete** and **insert**.

Dijkstra's Algorithm using Priority Queues

```
Q ⇐ makePQ()
insert(Q, (s, 0))
for each node u ≠ s do
    insert(Q, (u,∞))
S ⇐ ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        decreaseKey([)]Q, (u, min(dist(s, u), dist(s, v) + ℓ(v, u))).
```

Priority Queue operations:

1. $O(n)$ **insert** operations
2. $O(n)$ **extractMin** operations
3. $O(m)$ **decreaseKey** operations

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

1. All operations can be done in $O(\log n)$ time

Dijkstra's algorithm can be implemented in $O((n + m) \log n)$ time.

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

1. All operations can be done in $O(\log n)$ time

Dijkstra's algorithm can be implemented in $O((n + m) \log n)$ time.

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **delete** in $O(\log n)$ time.
2. **insert** in $O(1)$ *amortized* time.
3. **decreaseKey** in $O(1)$ *amortized* time: ℓ **decreaseKey** operations for $\ell \geq n$ take *together* $O(\ell)$ time
4. Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

1. Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **delete** in $O(\log n)$ time.
2. **insert** in $O(1)$ *amortized* time.
3. **decreaseKey** in $O(1)$ *amortized* time: ℓ **decreaseKey** operations for $\ell \geq n$ take *together* $O(\ell)$ time
4. Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

1. Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **delete** in $O(\log n)$ time.
2. **insert** in $O(1)$ *amortized* time.
3. **decreaseKey** in $O(1)$ *amortized* time: ℓ **decreaseKey** operations for $\ell \geq n$ take *together* $O(\ell)$ time
4. Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

1. Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **delete** in $O(\log n)$ time.
2. **insert** in $O(1)$ *amortized* time.
3. **decreaseKey** in $O(1)$ *amortized* time: ℓ **decreaseKey** operations for $\ell \geq n$ take *together* $O(\ell)$ time
4. Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

1. Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to V.
Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s,0))
prev(s) ⇐ null
for each node u ≠ s do
    insert(Q, (u,∞) )
    prev(u) ⇐ null

S = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        if (dist(s,v) + ℓ(v,u) < dist(s,u) ) then
            decreaseKey(Q, (u, dist(s,v) + ℓ(v,u)) )
            prev(u) = v
```

Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to V.
Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s,0))
prev(s) ⇐ null
for each node u ≠ s do
    insert(Q, (u,∞) )
    prev(u) ⇐ null

S = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        if (dist(s,v) + ℓ(v,u) < dist(s,u) ) then
            decreaseKey(Q, (u, dist(s,v) + ℓ(v,u)) )
            prev(u) = v
```

Shortest Path Tree

Lemma

The edge set $(u, \mathrm{prev}(u))$ *is the* reverse *of a shortest path tree rooted at* s*. For each* u*, the reverse of the path from* u *to* s *in the tree is a shortest path from* s *to* u*.*

Proof Sketch.

1. The edge set $\{(u, \mathrm{prev}(u)) \mid u \in V\}$ induces a directed in-tree rooted at s (Why?)
2. Use induction on $|S|$ to argue that the tree is a shortest path tree for nodes in V.

□

Shortest paths to s

Dijkstra's algorithm gives shortest paths from s to all nodes in V.
How do we find shortest paths from all of V to s?

1. In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
2. In directed graphs, use Dijkstra's algorithm in G^{rev}!

Shortest paths to s

Dijkstra's algorithm gives shortest paths from s to all nodes in V.
How do we find shortest paths from all of V to s?

1. In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
2. In directed graphs, use Dijkstra's algorithm in G^{rev}!

Notes

Notes

Notes

Notes