

# HW 11 (due Monday, Noon, May 4, 2015)

OLD CS 473: Fundamental Algorithms, Spring 2015

Version: 1.02

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**Collaboration Policy:** For this homework, every student should do this homework on their own. Submissions should be done individually.

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**We will not provide a solution for this homework. Also, we do not provide the latex file for this homework.**

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## 1. (10 PTS.) Multiple choice

For each of the questions below choose the most appropriate answer. No **IDK** credit for this question!

- (A) Given a graph  $G$ . Deciding if there is an independent set  $X$  in  $G$ , such that  $G \setminus X$  (i.e., the graph  $G$  after we remove the vertices of  $X$  from it) is bipartite can be solved in polynomial time.

False:  True:  Answer depends on whether  $P = NP$ :

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- (B) Consider any two problems  $X$  and  $Y$  both of them in **NP**. There always exists a polynomial time reduction from  $X$  to  $Y$ .

False:  True:  Answer depends on whether  $P = NP$ :

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- (C) Given a graph represented using adjacency lists, it can be converted into matrix representation in linear time in the size of the graph (i.e., linear in the number of vertices and edges of the graph).

False:  True:  Answer depends on whether  $P = NP$ :

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- (D) Given a **2SAT** formula  $F$ , there is always an assignment to its variables that satisfies at least  $(7/8)m$  of its clauses. False:  True:  Answer depends on whether  $P = NP$ :
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- (E) Given a graph  $G$ , deciding if contains a clique made out of 165 vertices is **NP-COMplete**. False:  True:  Answer depends on whether  $P = NP$ :
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- (F) Given a network flow  $G$  with lower bounds and capacities on the edges (say all numbers are integers that are smaller than  $n$ ). Assume  $f$  and  $g$  are two different maximum flows in  $G$  that complies with the lower bounds and capacity constraints. Then, the flow  $0.7f + 0.3g$  is always a valid maximum flow in  $G$ . False:  True:
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- (G) Given a directed graph  $G$  with positive weights on the edges, and a number  $k$ , finding if there is simple path in  $G$  from  $s$  to  $t$  (two given vertices of  $G$ ) with weight  $\geq k$ , can be done in polynomial time.

False:  True:  Answer depends on whether  $P = NP$ :

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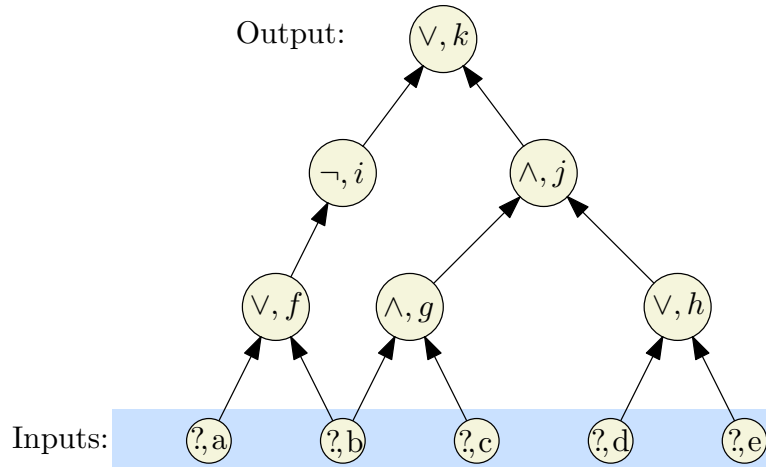
- (H) Given a directed graph  $G$  with (positive or negative) weights on its edges, computing the shortest walk from  $s$  to  $t$  in  $G$  can be done in polynomial time. False:  True:  Answer depends on whether  $P = NP$ :
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## 2. (10 PTS.) Short Questions.

- (A) (8 PTS.) Give a tight asymptotic bound for each of the following recurrences.

(I) (4 PTS.)  $A(n) = A(n - 3 \lceil \log n \rceil) + A(\lceil \log n \rceil) + \log n$ , for  $n > 2$   
and  $A(1) = A(2) = 1$ .

- (II) (4 PTS.)  $B(n) = 12B(\lfloor n/4 \rfloor) + B(\lfloor n/2 \rfloor) + n^2$ , for  $n > 10$  and  $B(i) = 1$  for  $1 \leq i \leq 10$ .
- (B) (8 PTS.) Convert the following boolean circuit (i.e., an instance of **Circuit-SAT**) into a CNF formula (i.e., an instance of SAT) such that the resulting formula is satisfiable if and only if the circuit sat instance is satisfiable. Use  $x_a, x_b, x_c, x_d, \dots$  as the variable names for the corresponding gates in the drawing. (You may need additional variables.) Note, that a node  $(\wedge, g)$  in the figure below denotes an and gate, where  $g$  is its label.



### 3. (20 PTS.) Balancing vampires.

Sadly, there are  $n$  vampires  $p_1, \dots, p_n$  in Champaign. The  $i$ th vampire has a score  $w_i \geq 0$  describing how well it can climb mountains. You want to divide the vampires into two teams, and you want the division of teams to be as fair as possible. The score of a team is the sum of the scores of all the vampires in that team. We want to minimize the differences of the scores of the two teams. Assume that for all  $i$ ,  $w_i \leq W$ .

- (A) (10 PTS.) Given integers  $\alpha, \beta \geq 0$ , and  $T_\alpha, T_\beta$ , such that  $\alpha + \beta = n$ , describe an algorithm, as fast as possible, to compute the partition into two teams, such that the first team has  $\alpha$  players of total score  $T_\alpha$ , and the second team has  $\beta$  players with total score  $T_\beta$ . What is the running time of your algorithm? (For any credit, it has to be polynomial in  $n$  and  $W$ .)  
(To simplify things, you can solve the decision version problem first, and describe shortly how to modify it to yield the desired partition.)
- (B) (5 PTS.) Describe an algorithm, as fast as possible, to compute the scores of the two teams in an optimal division that is as balanced as possible, when requiring that the two teams have exactly the same number of players (assume  $n$  is even). What is the running time of your algorithm?
- (C) (5 PTS.) State **formally** the decision version of the problem in (B), and prove that it is **NP-COMplete**. (There are several possible solutions for this part – pick the one you find most natural. Note, that the teams must have the same number of players.)

### 4. (12 PTS.) MAX Cut and MAX 2SAT.

The **Max CUT** problem is the following:

#### MAX Cut

**Instance:** Undirected graph  $G$  with  $n$  vertices and  $m$  edges, and an integer  $k$ .  
**Question:** Is there an undirected cut in  $G$  that cuts at least  $k$  edges?

#### MAX 2SAT

**Instance:** A 2CNF formula  $F$ , and an integer  $k$ .  
**Question:** Is there a truth assignment in  $F$  that satisfies at least  $k$  clauses.

You are given that **MAX Cut** is **NP-COMplete**. Prove that **MAX 2SAT** is **NP-COMplete** by a reduction

to/from **MAX Cut** (be sure to do the reduction in the right direction! [and please do not ask us what is the right direction – this is part of the problem]).

Hint: Think about how to encode a cut, by associating a boolean variable with each vertex of the graph. It might be a good idea to verify your answer by considering a graph with two vertices and a single edge between them and checking all possibilities for this case.

**5.** (10 PTS.) **Billboards are forever.**

Consider a stretch of Interstate-57 that is  $m$  miles long. We are given an ordered list of mile markers,  $x_1, x_2, \dots, x_n$  in the range  $0$  to  $m$ , at each of which we are allowed to construct billboards (suppose they are given as an array  $X[1 \dots n]$ ). Suppose we can construct billboards for free, and that we are given an array  $R[1 \dots n]$ , where  $R[i]$  is the revenue we would receive by constructing a billboard at location  $X[i]$ . Given that state law requires billboards to be at least 5 miles apart, describe an algorithm, as fast as possible, to compute the maximum revenue we can acquire by constructing billboards.

What is the running time of your algorithm? (For full credit, your algorithm has to be as fast as possible.)

**6.** (10 PTS.) **Best edge ever.**

You are given a directed graph  $G$  with  $n$  vertices and  $m$  edges. For every edge  $e \in E(G)$ , there is an associated weight  $w(e) \in \mathbb{R}$ . For a path (not necessarily simple)  $\pi$  in  $G$ , its **quality** is  $W(\pi) = \max_{e \in \pi} w(e)$ . We are interested in computing the highest quality walk in  $G$  between two given vertices (say  $s$  and  $t$ ). Either **prove** that computing such a walk is **NP-HARD**, or alternatively, provide an algorithm (and **prove** its correctness) for this problem (the algorithm has to be as fast as possible – what is the running time of your algorithm?).

**7.** (10 PTS.) **Dominate this.**

You are given a set of intervals  $\mathcal{I} = \{I_1, \dots, I_n\}$  on the real line (assume all with distinct endpoints) – they are given in arbitrary order (i.e., you can not assume anything on the ordering). Consider the problem of finding a set of intervals  $\mathcal{K} \subseteq \mathcal{I}$ , as small as possible, that dominates all the other intervals. Formally,  $\mathcal{K}$  **dominates**  $\mathcal{I}$ , if for every interval  $I \in \mathcal{I}$ , there is an interval  $K \in \mathcal{K}$ , such that  $I$  intersects  $K$ .

Describe an algorithm (as fast as possible) that computes such a minimal dominating set of  $\mathcal{I}$ . What is the running time of your algorithm? Prove the correctness of your algorithm.

**8.** (10 PTS.) **Network flow.**

You are given a network flow  $G$  (with integer capacities on the edges, and a source  $s$  and a sink  $t$ ), and a maximum flow  $f$  on it (you can assume  $f$  is integral). You want increase the maximum flow in  $G$  by one unit by applying a single augmenting path to  $f$ . Naturally, to be able to do that, you must increase the capacity of some of the edges of  $G$ . In particular, for every edge  $e \in E(G)$ , there is an associated cost  $\text{cost}(e)$  of increasing its capacity by one unit. Describe an algorithm, that computes (as fast as possible), the cheapest collection of edges of  $G$ , such that if we increase the capacity on each of these edges by 1, then one can find an augmenting path to  $f$  that increases its flow by one unit. How fast is your algorithm?

Provide an argument that explains why your algorithm is correct.