HW 4 (due Monday, at noon, March 2, 2015)

OLD CS 473: Fundamental Algorithms, Spring 2015 Version: **1.01**

You also have to do the quiz online (on moodle).

Collaboration Policy: For this homework, Problems 1–3 can be worked in groups of up to three students.

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1. (30 PTS.) Contaminated vertices.

Consider a directed graph G with real weights on its edges. A vertex v is ***contaminated*** if there is a negative cycle in G (not necessary simple) that includes v. Describe a polynomial time algorithm that computes all the contaminated vertices in G.

(Hint: First come up with an algorithm that extract the negative cycle in G if such a cycle exists.)

2. (30 PTS.) Is your overlap in vain?

Let $\mathcal{I}$ be a set of n closed intervals on the real line (assume they all have distinct endpoints). A set of intervals $\mathsf{X} \subseteq \mathcal{I}$ is ***admissible*** if no point on the real line is covered by more than two intervals of X. The ***profit*** of X is

$$\phi(\mathsf{X}) = \sum_{I,J \in \mathsf{X}, I \neq J} |I \cap J| ,$$

where $|I \cap J|$ is the length of the interval $I \cap J$.

Describe an algorithm, as fast as possible, that outputs the subset of $\mathcal{I}$ that maximizes the profit, and is admissible.

3. (40 PTS.) Rank and file.

(A) (10 PTS.) The ***rank*** of a vertex v in a DAG G, is the length of the longest path in DAG that starts in v. Describe a linear time algorithm (in the number of edges and vertices of G) that computes for all the vertices in G their rank.

(B) (5 PTS.) Prove that if two vertices $u, v \in \mathsf{V}(\mathsf{G})$ have the same rank (again, G is a DAG), then the edges (u, v) and (v, u) are not in G.

(C) (5 PTS.) Using (B), prove that in any DAG G with n vertices, for any k, either there is a path of length k, or there is a set X of $\lfloor n/k \rfloor$ vertices in G that is ***independent***; that is, there is no edge between any pair of vertices of X.

(D) (10 PTS.) Consider a set of P of n points in the plane. The points of P are in general position – no two points have the same x or y coordinates. Consider a sequence S of points $\mathsf{p}_1, \mathsf{p}_2, \ldots, \mathsf{p}_k$ of P, where $\mathsf{p}_i = (x_i, y_i)$, for $i = 1, \ldots, k$. The sequence S is ***diagonal***, if either

- for all $i = 1, \ldots, k-1$, we have $x_i < x_{i+1}$ and $y_i < y_{i+1}$, or
- for all $i = 1, \ldots, k-1$, we have $x_i < x_{i+1}$ and $y_i > y_{i+1}$.

Prove using (C) that there is always a diagonal of length $\lfloor \sqrt{n} \rfloor$ in P. Describe an algorithm, as fast as possible, that computes the longest diagonal in P.

(E) (10 PTS.) Using the algorithm of (D), describe a polynomial time algorithm that decomposes P into a set of $O(\sqrt{n})$ disjoint diagonals. Prove the correctness of your algorithm.