

co-NP, Self-Reductions

Lecture 24

April 29, 2014

Find this path

G : A directed graph G with weights on edges, vertices s and t , and a number $\Delta \geq 0$. Consider the following problems, that all ask if there exists a path π between s and t with a desired property.

We have that

- (A) **ShortestP** \in **P**, **ExactP** \in **P**, **LongestP** \in **P**.
- (B) **ShortestP** \in **P**, **ExactP** \in **P**, **LongestP** \in **NPC**.
- (C) **ShortestP** \in **NPC**, **ExactP** \in **P**, **LongestP** \in **P**.
- (D) **ShortestP** \in **P**, **ExactP** \in **NPC**, **LongestP** \in **NPC**.
- (E) **ShortestP** \in **NPC**, **ExactP** \in **NPC**, **LongestP** \in **NPC**.

- 1 **ShortestP**:
 $\text{length}(\pi) \leq \Delta$.
- 2 **ExactP**:
 $\text{length}(\pi) = \Delta$.
- 3 **LongestP**:
 $\text{length}(\pi) \geq \Delta$.

Part I

Complementation and Self-Reduction

The class **P**

- 1 A language **L** (equivalently decision problem) is in the class **P** if there is a polynomial time algorithm **A** for deciding **L**; that is given a string **x**, **A** correctly decides if $x \in L$ and running time of **A** on **x** is polynomial in $|x|$, the length of **x**.

The class **NP**

Two equivalent definitions:

- 1 Language **L** is in **NP** if there is a non-deterministic polynomial time algorithm **A** (Turing Machine) that decides **L**.
 - 1 For $x \in L$, **A** has some non-deterministic choice of moves that will make **A** accept x
 - 2 For $x \notin L$, no choice of moves will make **A** accept x
- 2 **L** has an efficient certifier **C**(\cdot, \cdot).
 - 1 **C** is a polynomial time deterministic algorithm
 - 2 For $x \in L$ there is a string **y** (proof) of length polynomial in $|x|$ such that **C**(x, y) accepts
 - 3 For $x \notin L$, no string **y** will make **C**(x, y) accept

Short certificates...

Let X be an **NP-Complete** problem, such that for any YES instance I of X of length n , there exists a certificate of length $O(\log n)$ that it is correct.

If such a problem X exists then

- (A) Nothing at all is implied by this.
- (B) $P = NP$.
- (C) $P \neq NP$.

Complementation

Definition

Given a decision problem X , its **complement** \bar{X} is the collection of all instances s such that $s \notin L(X)$

Equivalently, in terms of languages:

Definition

Given a language L over alphabet Σ , its **complement** \bar{L} is the language $\Sigma^* \setminus L$.

Examples

- $\text{PRIME} = \{n \mid n \text{ is an integer and } n \text{ is prime}\}$
 $\overline{\text{PRIME}} = \{n \mid n \text{ is an integer and } n \text{ is not a prime}\}$
 $\overline{\text{PRIME}} = \text{COMPOSITE.}$
- $\text{SAT} = \{\varphi \mid \varphi \text{ is a CNF formula and } \varphi \text{ is satisfiable}\}$
 $\overline{\text{SAT}} = \{\varphi \mid \varphi \text{ is a CNF formula and } \varphi \text{ is not satisfiable}\}.$
 $\overline{\text{SAT}} = \text{UnSAT.}$

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- ② $\text{SAT} = \{\varphi \mid \varphi \text{ is a CNF formula and } \varphi \text{ is satisfiable}\}$
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Technicality: $\overline{\text{SAT}}$ also includes strings that do not encode any valid CNF formula. Typically we ignore those strings because they are not interesting. In all problems of interest, we assume that it is “easy” to check whether a given string is a valid instance or not.

P is closed under complementation

Proposition

Decision problem X is in P if and only if \bar{X} is in P .

Proof.

- 1 If X is in P let A be a polynomial time algorithm for X .
- 2 Construct polynomial time algorithm A' for \bar{X} as follows: given input x , A' runs A on x and if A accepts x , A' rejects x and if A rejects x then A' accepts x .
- 3 Only if direction is essentially the same argument.



Asymmetry of NP

Definition

Nondeterministic Polynomial Time (denoted by **NP**) is the class of all problems that have efficient certifiers.

Observation

To show that a problem is in **NP** we only need short, efficiently checkable certificates for “yes”-instances. What about “no”-instances?

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Given a **CNF** formula φ , is φ unsatisfiable?

Easy to give a proof that φ is satisfiable (an assignment) but no easy (known) proof to show that φ is unsatisfiable!

Examples of complement problems

Some languages

- 1 **UnSAT**: CNF formulas φ that are not satisfiable
- 2 **No-Hamilton-Cycle**: graphs G that do not have a Hamilton cycle
- 3 **No-3-Color**: graphs G that are not 3-colorable

Examples of complement problems

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Above problems are complements of known **NP** problems (viewed as languages).

NP and co-NP

NP

Decision problems with a polynomial certifier.

Examples: **SAT**, **Hamiltonian Cycle**, **3-Colorability**.

Definition

co-NP is the class of all decision problems **X** such that $\bar{X} \in \text{NP}$.

Examples: **UnSAT**, **No-Hamiltonian-Cycle**, **No-3-Colorable**.

If L is a language in **co-NP** then that there is a polynomial time certifier/verifier $C(\cdot, \cdot)$, such that:

- 1 for $s \notin L$ there is a proof t of size polynomial in $|s|$ such that $C(s, t)$ correctly says NO.
- 2 for $s \in L$ there is no proof t for which $C(s, t)$ will say NO

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co-NP has checkable proofs for strings NOT in the language.

Natural Problems in **co-NP**

- 1 **Tautology**: given a Boolean formula (not necessarily in **CNF** form), is it true for *all* possible assignments to the variables?
- 2 **Graph expansion**: given a graph **G**, is it an *expander*? A graph **G = (V, E)** is an **expander** if and only if for each **S ⊂ V** with $|S| \leq |V|/2$, $|N(S)| \geq |S|$. Expanders are very important graphs in theoretical computer science and mathematics.

Factorization, Primality

Problem: Primality

Instance: An integer n .

Question: Is the number n prime?

Problem: Factoring

Instance: Integers n, k .

Question: Does the number n have a factor $\leq k$?
Formally, is there l , such that $2 \leq l \leq k$, such that l divides n ?

- 1 **Primality** is in **P**.
- 2 **Factoring** is in **NP** \cap **co-NP**.

Factoring is a very naughty problem

Problem: **Factoring**

Instance: Integers n, k .

Question: Does the number n has a factor $\leq k$? Formally, is there ℓ , such that $2 \leq \ell \leq k$, such that ℓ divides n ?

If answer is:

- 1 NO: certificate is all prime factors of n . Certification: multiply the given numbers.
- 2 YES: Certificate is the factor ℓ . Verify it divides n .

Belief: Unlikely **Factoring** is **NP-Complete**. Can be solved in polynomial time on a quantum computer.

P, NP, co-NP

co-P: complement of P. Language **X** is in **co-P** iff $\bar{X} \in P$

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$P = \text{co-P}$.

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$P \subseteq NP \cap \text{co-NP}$.

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Proposition

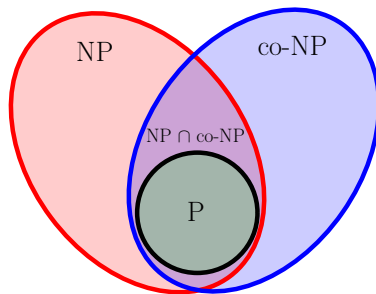
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Proposition

$P \subseteq NP \cap \text{co-NP}$.

Saw that $P \subseteq NP$. Same proof shows $P \subseteq \text{co-NP}$.

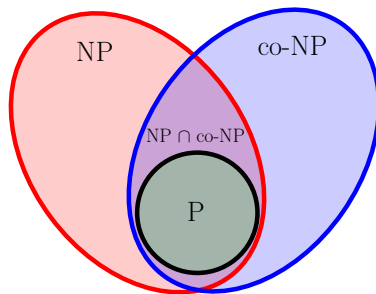
P, NP, and co-NP



Open Problems:

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P, NP, and co-NP

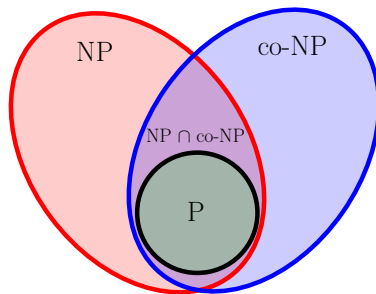


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Consensus opinion: No.

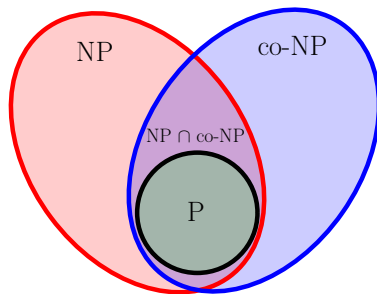
P, NP, and co-NP



Open Problems:

- 1 Does $\mathbf{NP} = \mathbf{co-NP}$?
Consensus opinion: No.
- 2 Is $\mathbf{P} = \mathbf{NP} \cap \mathbf{co-NP}$?

P, NP, and co-NP



Open Problems:

- 1 Does $\mathbf{NP} = \mathbf{co-NP}$?
Consensus opinion: No.
- 2 Is $\mathbf{P} = \mathbf{NP} \cap \mathbf{co-NP}$?
No real consensus.

P, NP, and co-NP

Proposition

If $P = NP$ then $NP = \text{co-NP}$.

P, NP, and co-NP

Proposition

If $P = NP$ then $NP = \text{co-NP}$.

Proof.

$P = \text{co-P}$

If $P = NP$ then $\text{co-NP} = \text{co-P} = P$. □

P, NP, and co-NP

We know that:

Proposition

If $P = NP$ then $NP = \text{co-NP}$.

As such:

- (A) If $P \neq NP$ then $NP \neq \text{co-NP}$.
- (B) If $NP \neq \text{co-NP}$ then $P \neq NP$.
- (C) If $NP = \text{co-NP}$ then $P \neq NP$.
- (D) If $NP \neq \text{co-NP}$ then $P = NP$.
- (E) None of the above.

P, NP, and co-NP

Which means that...

Corollary

If $\text{NP} \neq \text{co-NP}$ then $\text{P} \neq \text{NP}$.

P, NP, and co-NP

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Corollary

If $\text{NP} \neq \text{co-NP}$ then $\text{P} \neq \text{NP}$.

Importance of corollary: try to prove $\text{P} \neq \text{NP}$ by proving that $\text{NP} \neq \text{co-NP}$.

Complexity Class NP \cap co-NP

Problems in this class have

- 1 Efficient certifiers for yes-instances
- 2 Efficient disqualifiers for no-instances

Problems have a **good characterization** property, since for both yes and no instances we have short efficiently checkable proofs.

NP \cap co-NP: Example

Example

Bipartite Matching: Given bipartite graph $G = (U \cup V, E)$, does G have a perfect matching?

Bipartite Matching \in NP \cap co-NP

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- 1 If G is a yes-instance, then proof is just the perfect matching.

Example (More interesting...)

Factoring \in NP \cap co-NP, and we do not know if it is in P!

NP \cap co-NP: Example

Example

Bipartite Matching: Given bipartite graph $G = (U \cup V, E)$, does G have a perfect matching?

Bipartite Matching \in NP \cap co-NP

- 1 If G is a yes-instance, then proof is just the perfect matching.
- 2 If G is a no-instance, then by Hall's Theorem, there is a subset of vertices $A \subseteq U$ such that $|N(A)| < |A|$.

Example (More interesting...)

Factoring \in NP \cap co-NP, and we do not know if it is in P!

Good Characterization $\stackrel{?}{=}$ Efficient Solution

- 1 Bipartite Matching has a polynomial time algorithm
- 2 Do all problems in $\mathbf{NP} \cap \mathbf{co-NP}$ have polynomial time algorithms? That is, is $\mathbf{P} = \mathbf{NP} \cap \mathbf{co-NP}$?

Good Characterization $\stackrel{?}{=}$ Efficient Solution

- 1 Bipartite Matching has a polynomial time algorithm
- 2 Do all problems in $\mathbf{NP} \cap \mathbf{co-NP}$ have polynomial time algorithms? That is, is $\mathbf{P} = \mathbf{NP} \cap \mathbf{co-NP}$?
Problems in $\mathbf{NP} \cap \mathbf{co-NP}$ have been proved to be in \mathbf{P} many years later
 - 1 Linear programming (Khachiyan 1979)
 - 1 Duality easily shows that it is in $\mathbf{NP} \cap \mathbf{co-NP}$
 - 2 Primality Testing (Agarwal-Kayal-Saxena 2002)
 - 1 Easy to see that \mathbf{PRIME} is in $\mathbf{co-NP}$ (why?)
 - 2 \mathbf{PRIME} is in \mathbf{NP} - not easy to show! (Vaughan Pratt 1975)

$P \stackrel{?}{=} NP \cap \text{co-NP}$ (contd)

- 1 Some problems in $NP \cap \text{co-NP}$ still cannot be proved to have polynomial time algorithms
 - 1 Parity Games.
 - 2 Other more specialized problems.

Definition

A problem X is said to be **co-NP-Complete** (**co-NPC**) if

- 1 $X \in \text{co-NP}$
- 2 (**Hardness**) For any $Y \in \text{co-NP}$, $Y \leq_P X$

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co-NP-Complete problems are the hardest problems in **co-NP**.

co-NP Completeness

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Lemma

X is **co-NPC** if and only if \bar{X} is **NP-Complete**.

Proof left as an exercise.

coNP-Complete?

Every **co-NP**-Complete problem has an exponential time algorithm; that is, an algorithm that runs in $2^{p(n)}$ time on an input of size n for some fixed polynomial $p(\cdot)$.

- (A) True.
- (B) False.
- (C) True only if $P = NP$.
- (D) True only if $P \neq NP$.

P, NP and co-NP

Possible scenarios:

- 1 $P = NP$. Then $P = NP = \text{co-NP}$.
- 2 $NP = \text{co-NP}$ and $P \neq NP$ (and hence also $P \neq \text{co-NP}$).
- 3 $NP \neq \text{co-NP}$. Then $P \neq NP$ and also $P \neq \text{co-NP}$.

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Theorem (Ladner)

*If $P \neq NP$ then there is a problem/language $X \in NP \setminus P$ such that X is not **NP-Complete**.*

Karp vs Turing Reduction and NP vs co-NP

Question: Why restrict to Karp reductions for NP-Completeness?

Lemma

If $X \in \text{co-NP}$ and Y is **NP-Complete** then $X \leq_P Y$ under Turing reduction.

Thus, Turing reductions cannot distinguish **NP** and **co-NP**.

Back to Decision versus Search

- 1 Recall, decision problems are those with yes/no answers, while search problems require an explicit solution for a yes instance

Example

- 1 Satisfiability
 - 1 **Decision:** Is the formula φ satisfiable?
 - 2 **Search:** Find assignment that satisfies φ
- 2 Graph coloring
 - 1 **Decision:** Is graph G 3-colorable?
 - 2 **Search:** Find a 3-coloring of the vertices of G

Decision “reduces to” Search

- 1 Efficient algorithm for search implies efficient algorithm for decision.
- 2 If decision problem is difficult then search problem is also difficult.
- 3 Can an efficient algorithm for decision imply an efficient algorithm for search?

Yes, for all the problems we have seen. In fact for all **NP-Complete** Problems.

Self Reduction

Definition

A problem is said to be **self reducible** if the search problem reduces (by Turing reduction) in polynomial time to decision problem. In other words, there is an algorithm to solve the search problem that has polynomially many steps, where each step is either

- 1 A conventional computational step, or
- 2 a call to subroutine solving the decision problem.

Problem: MAX 3SAT

Instance: **F**: 3CNF Formula with **n** variables and **m** clauses.

Question: Find assignment satisfying as many clauses of **F** as possible.

- (A) MAX 3SAT is NP-Complete.
- (B) MAX 3SAT is NP-Hard.
- (C) MAX 3SAT is P.
- (D) MAX 3SAT is not a decision problem.

Proposition

SAT *is self reducible.*

In other words, there is a polynomial time algorithm to find the satisfying assignment if one can periodically check if some formula is satisfiable.

Search Algorithm for SAT

given a Decision Algorithm for SAT

Input: **SAT** formula φ with n variables x_1, x_2, \dots, x_n .

- 1 set $x_1 = 0$ in φ and get new formula φ_1 . check if φ_1 is satisfiable using decision algorithm. if φ_1 is satisfiable, recursively find assignment to x_2, x_3, \dots, x_n that satisfy φ_1 and output $x_1 = 0$ along with the assignment to x_2, \dots, x_n .
- 2 if φ_1 is not satisfiable then set $x_1 = 1$ in φ to get formula φ_2 . if φ_2 is satisfiable, recursively find assignment to x_2, x_3, \dots, x_n that satisfy φ_2 and output $x_1 = 1$ along with the assignment to x_2, \dots, x_n .
- 3 if φ_1 and φ_2 are both not satisfiable then φ is not satisfiable.

Algorithm runs in polynomial time if the decision algorithm for **SAT** runs in polynomial time. At most $2n$ calls to decision algorithm.

Self-Reduction for **NP-Complete** Problems

Theorem

Every **NP-Complete** problem/language L is self-reducible.

Proof is not hard but requires understanding of proof of Cook-Levin theorem.

Note that proof is only for complete languages, not for all languages in **NP**. Otherwise **Factoring** would be in polynomial time and we would not rely on it for our current security protocols.

Easy and instructive to prove self-reducibility for specific **NP-Complete** problems such as **Independent Set**, **Vertex Cover**, **Hamiltonian Cycle**, etc.

See discussion section problems.

Notes