HW 7 (due Tuesday, at noon, April 1st, 2014)

CS 473: Fundamental Algorithms, Spring 2014 Version: **1.0**

Make sure that you write the solutions for the problems on separate sheets of paper. Write your name and netid on each sheet.

Collaboration Policy: The homework can be worked in groups of up to 3 students each.

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1. (30 PTS.) Bottleneck Spanning Tree.
Let $G = (V, E)$ an undirected edge-weighted graph. The bottleneck weight of a spanning tree T is the weight of the maximum weight edge in T.

- Prove that the minimum spanning tree of a graph is also a spanning tree which minimizes the bottleneck weight.
- Describe an algorithm to compute a spanning tree with minimum bottleneck weight in linear time (that is, $O(|V| + |E|)$ time). No proof of correctness necessary but you need to justify the running time. *Hint:* Start by computing the median of the edge weights. Do you see how you can use this same idea to compute the bottleneck shortest path in linear time?

2. (40 PTS.) Dominating Sets and Domatic Partition.
Let $G = (V, E)$ be an undirected graph. Recall that $S \subseteq V$ is a dominating set if the following property holds: for every $u \in V$ we have $u \in S$ or some neighbor of u is in S. In the *domatic partition* problem we are given a graph G and the goal is to find a maximum number of mutually disjoint dominating sets in G. Let δ be the degree of a minimum degree node in G. It is easy to see that the domatic number is at most $(\delta + 1)$ since each dominating set has to contain u or some neighbor of u where u is a node with degree δ. In this problem we will see that the domatic number of a graph on n nodes and minimum degree δ is at least as large as $\lceil \frac{\delta+1}{c \ln n} \rceil$ for some sufficient large universal constant c. Note that this guarantees only 1 dominating set if $\delta < c \ln n$ (the entire vertex set can be chosen as the dominating set). Let $k = \lceil \frac{\delta+1}{c \ln n} \rceil$. Consider the following randomized algorithm. For each node u independently assign a color $g(u)$ that is chosen uniformly at random from the colors $\{1, 2, \ldots, k\}$.

(A) (20 pts) For a fixed node v and a fixed color i show that with probability at least $1 - 1/n^2$ there is a node with color i that is either v or a neighbor of v. Choose c sufficiently large to ensure this.

(B) (10 pts) Using the above show that for a fixed color i the set of nodes that are colored i form a dominating set for G with probability at least $1 - 1/n$.

(C) (10 pts) Using the above two parts argue that the domatic number of G is at least k.

The simple union bound is useful for this problem: for any set of events $A_1, A_2, \ldots, A_h$ we have $Pr[A_1 \cup A_2 \cup \ldots A_h] \le \sum_{i=1}^{h} Pr[A_i]$.

3. (30 PTS.) Hiring via Random Permutation.
A company has to hire from n potential candidates. They have to interview the candidates sequentially and either accept or reject immediately after the interview. The company, when evaluating a candidate can rank the candidate in comparision to all previously interviewed candidates (assume for simplicity that there are no ties in the ranking). They decide to adopt the following hiring policy. They pick a uniform random permutation of the candidates and

interview them in the order of the permutation. After interviewing the i'th candidate in the permutation, they hire that candidate if and only if he/she is the best among the first i candidates they have seen. For a fixed i what is the probability that the i'th candidate is hired? What is the expected number of candidates hired?