Applications of Network Flows

Lecture 18
March 28, 2013

Network Flow: Facts to Remember

Flow network: directed graph G, capacities c, source s, sink t.

1. Maximum s-t flow can be computed:
    1. Using Ford-Fulkerson algorithm in $O(mC)$ time when capacities are integral and C is an upper bound on the flow.
    2. Using variant of algorithm, in $O(m^2 \log C)$ time, when capacities are integral. (Polynomial time.)
    3. Using Edmonds-Karp algorithm, in $O(m^2 n)$ time, when capacities are rational (strongly polynomial time algorithm).

1. If capacities are integral then there is a maximum flow that is integral and above algorithms give an integral max flow. This is known as **integrality of flow**.

2. Given a flow of value v, can decompose into $O(m + n)$ flow paths of same total value v. Integral flow implies integral flow on paths.

3. Maximum flow is equal to the minimum cut and minimum cut can be found in $O(m + n)$ time given any maximum flow.

Paths, Cycles and Acyclicity of Flows

Definition

Given a flow network $G = (V, E)$ and a flow $f : E \to \mathbb{R}^{\geq 0}$ on the edges, the **support** of f is the set of edges $E' \subseteq E$ with non-zero flow on them. That is, $E' = \{e \in E \mid f(e) > 0\}$.
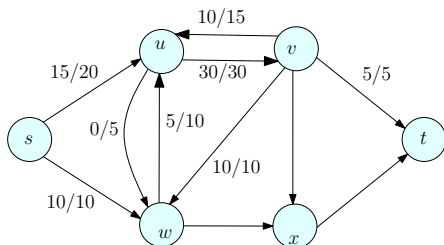
Question: Given a flow f, can there by cycles in its support?

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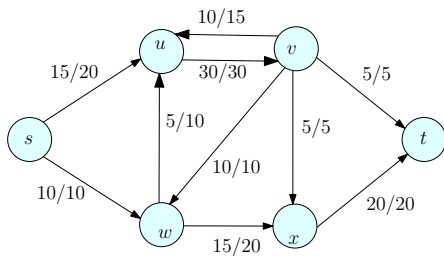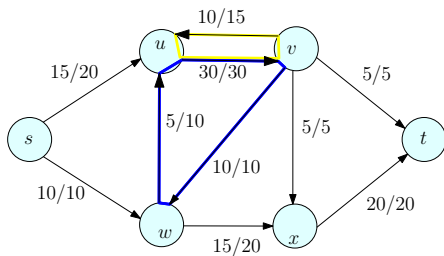Question: Given a flow $\mathbf{f}$, can there by cycles in its support?

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Acyclicity of Flows

Proposition

In any flow network, if $\mathbf{f}$ is a flow then there is another flow $\mathbf{f'}$ such that the support of $\mathbf{f'}$ is an acyclic graph and $\mathbf{v(f')} = \mathbf{v(f)}$. Further if $\mathbf{f}$ is an integral flow then so is $\mathbf{f'}$.
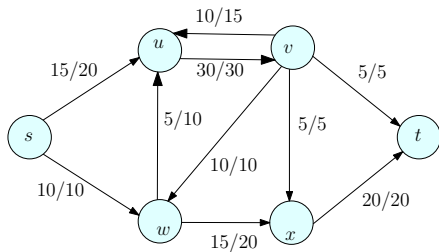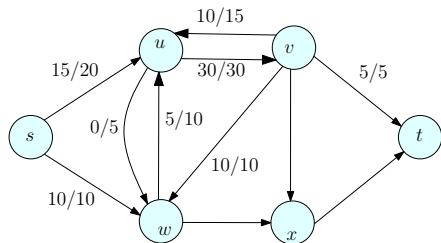
Proof.

1. $\mathbf{E'} = \{\mathbf{e} \in \mathbf{E} \mid \mathbf{f(e)} > \mathbf{0}\}$, support of $\mathbf{f}$.
2. Suppose there is a directed cycle $\mathbf{C}$ in $\mathbf{E'}$
3. Let $\mathbf{e'}$ be the edge in $\mathbf{C}$ with least amount of flow
4. For each $\mathbf{e} \in \mathbf{C}$, reduce flow by $\mathbf{f(e')}$. Remains a flow. Why?
5. Flow on $\mathbf{e'}$ is reduced to $\mathbf{0}$.
6. Claim: Flow value from $\mathbf{s}$ to $\mathbf{t}$ does not change. Why?
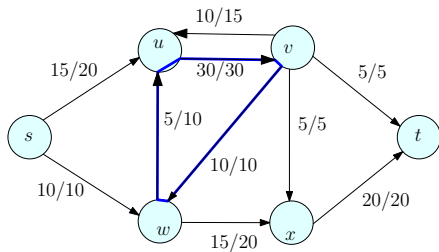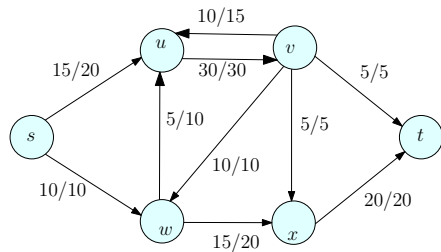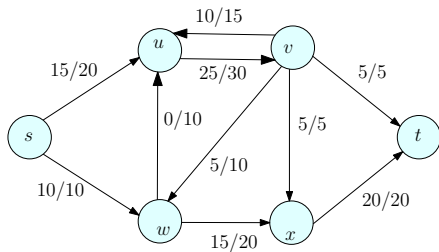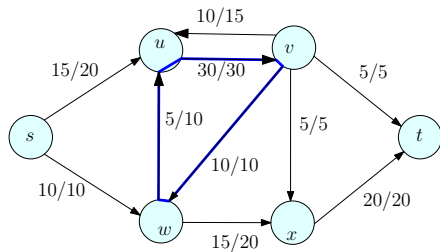7. Iterate until no cycles ☐

Example
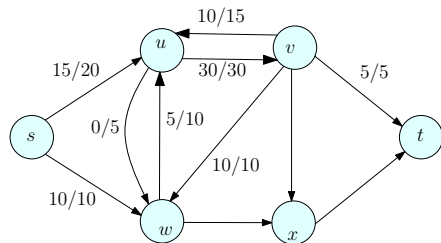
Example

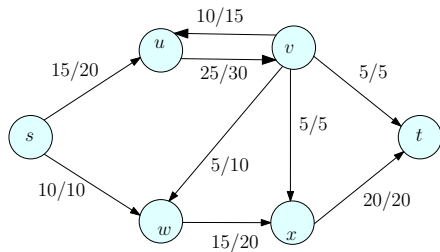
Throw away edge with no flow on it
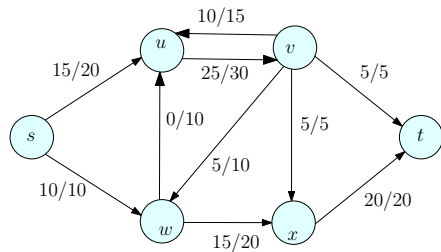
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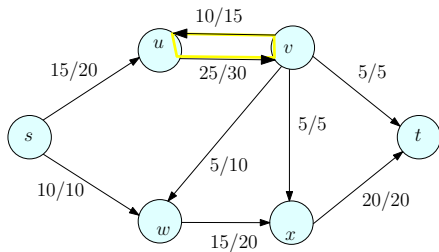
Find a cycle in the support/flow

Example

Reduce flow on cycle as much as possible

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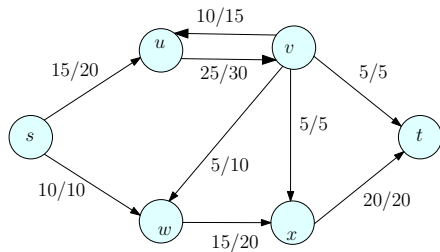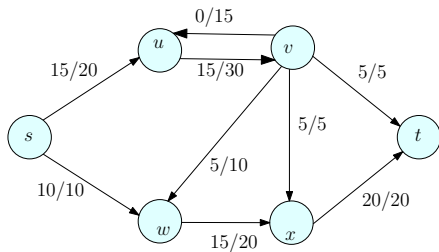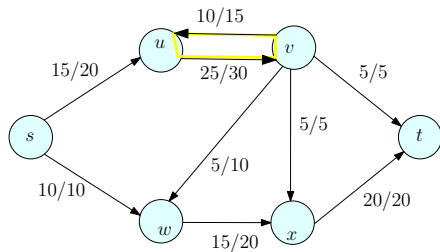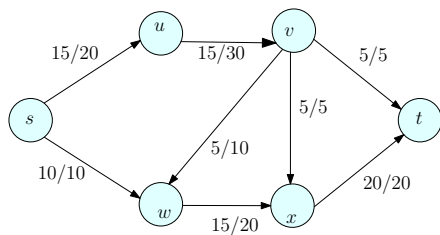
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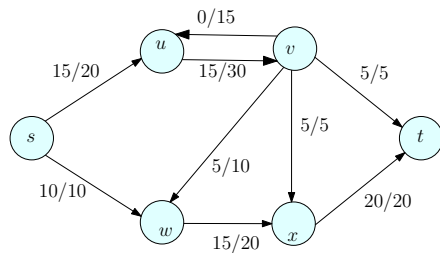
Example

Throw away edge with no flow on it

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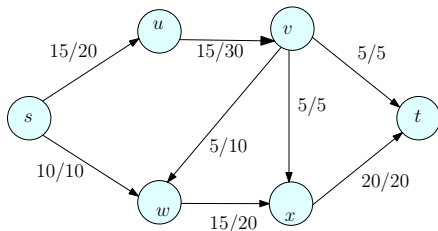
Viola!!! An equivalent flow with no cycles in it. Original flow:

Flow Decomposition

Lemma

Given an edge based flow $\mathbf{f} : \mathbf{E} \to \mathbb{R}^{\geq 0}$, there exists a collection of paths $\mathcal{P}$ and cycles $\mathcal{C}$ and an assignment of flow to them $\mathbf{f'} : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}^{\geq 0}$ such that:

1. *$|\mathcal{P} \cup \mathcal{C}| \leq \mathbf{m}$*
2. *for each $\mathbf{e} \in \mathbf{E}$, $\sum_{\mathbf{P} \in \mathcal{P} : \mathbf{e} \in \mathbf{P}} \mathbf{f'}(\mathbf{P}) + \sum_{\mathbf{C} \in \mathcal{C} : \mathbf{e} \in \mathbf{C}} \mathbf{f'}(\mathbf{C}) = \mathbf{f}(\mathbf{e})$*
3. *$\mathbf{v}(\mathbf{f}) = \sum_{\mathbf{P} \in \mathcal{P}} \mathbf{f'}(\mathbf{P})$.*
4. *if $\mathbf{f}$ is integral then so are $\mathbf{f'}(\mathbf{P})$ and $\mathbf{f'}(\mathbf{C})$ for all $\mathbf{P}$ and $\mathbf{C}$*

Proof Idea.

1. Remove all cycles as in previous proposition.
2. Next, decompose into paths as in previous lecture.
3. Exercise: verify claims.

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1. $|\mathcal{P} \cup \mathcal{C}| \leq m$
2. *for each* $e \in E$, $\sum_{P \in \mathcal{P} : e \in P} f'(P) + \sum_{C \in \mathcal{C} : e \in C} f'(C) = f(e)$
3. $v(f) = \sum_{P \in \mathcal{P}} f'(P)$.
4. *if* f *is integral then so are* $f'(P)$ *and* $f'(C)$ *for all* P *and* C

Proof Idea.

1. Remove all cycles as in previous proposition.
2. Next, decompose into paths as in previous lecture.
3. Exercise: verify claims. □

Example
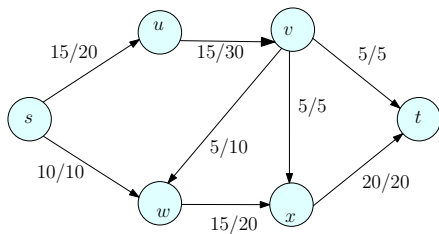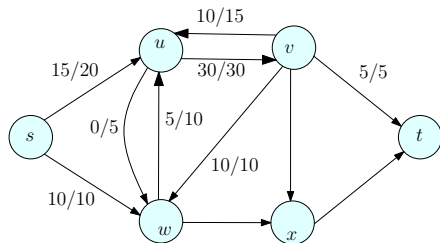
Find cycles as shown before

Example
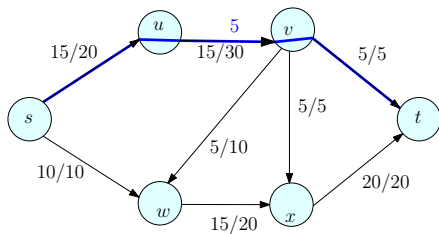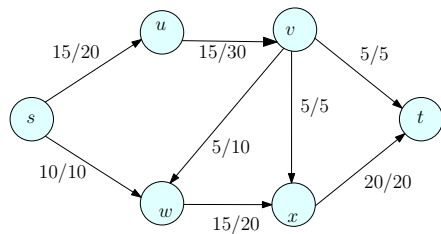
Find a source to sink path, and push max flow along it (5 unites)
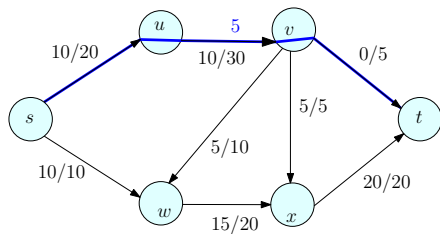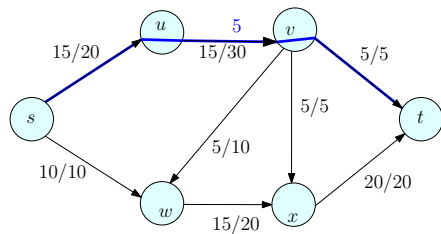
Example

Compute remaining flow
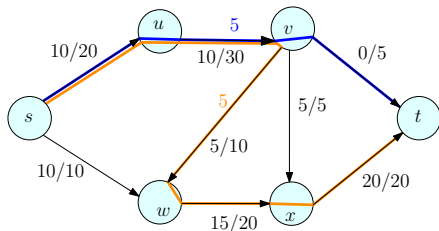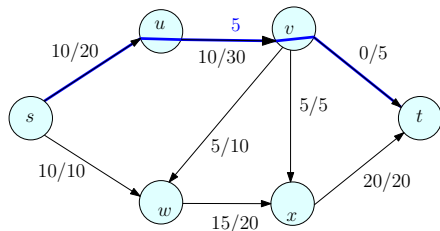
Example

Find a source to sink path, and push max flow along it (5 unites).
Edges with **0** flow on them can not be used as they are no longer in
the support of the flow.

Example
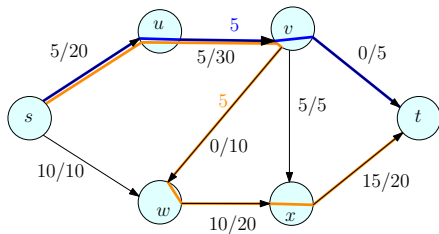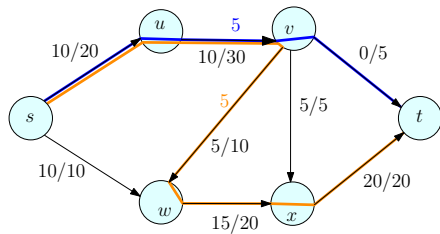
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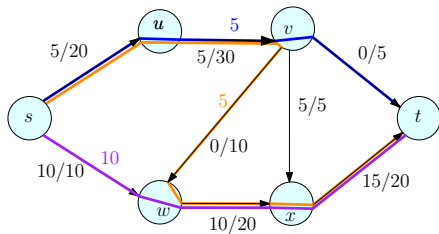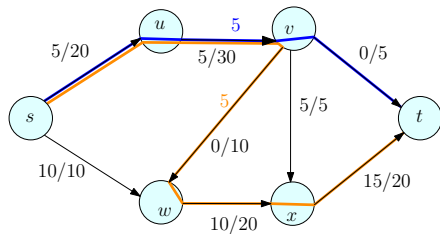
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Find a source to sink path, and push max flow along it (10 unites).

Example
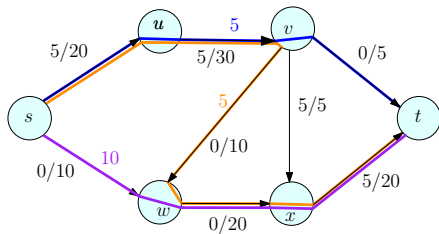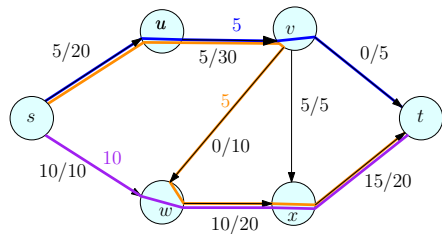
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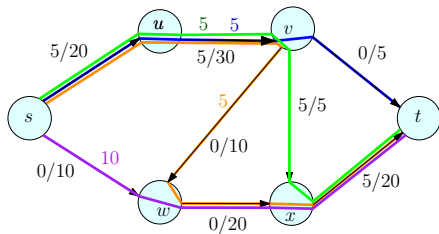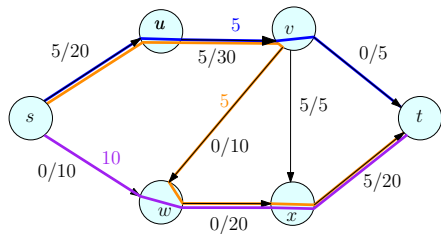
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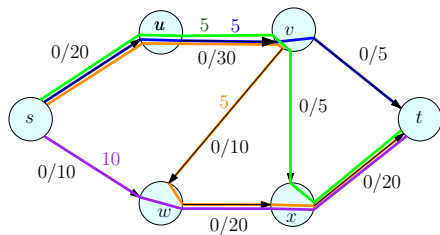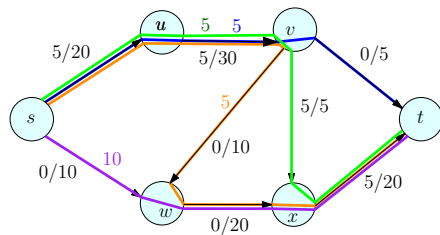
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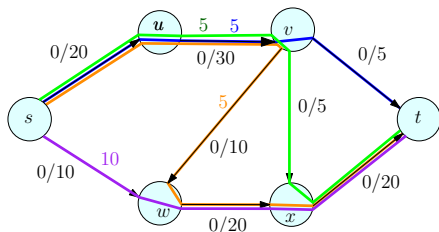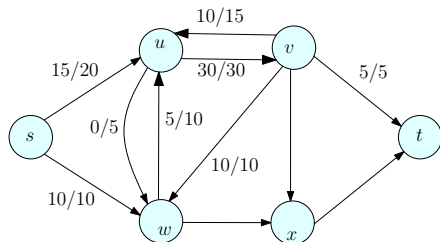
No flow remains in the graph. We fully decomposed the flow into flow on paths. Together with the cycles, we get a decomposition of the original flow into **m** flows on paths and cycles.

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Above flow decomposition can be computed in $\mathbf{O(m^2)}$ time.

Part I

Network Flow Applications I

Edge-Disjoint Paths in Directed Graphs
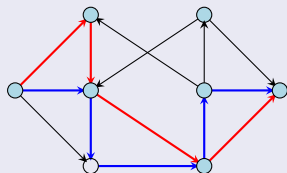
Definition

A set of paths is **edge disjoint** if no two paths share an edge.

Problem

Given a directed graph with two special vertices **s** and **t**, find the *maximum* number of edge disjoint paths from **s** to **t**.

Applications: Fault tolerance in routing — edges/nodes in networks can fail. Disjoint paths allow for planning backup routes in case of failures.

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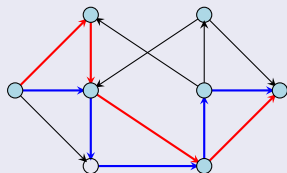
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Reduction to Max-Flow

Problem

Given a directed graph **G** with two special vertices **s** and **t**, find the maximum number of edge disjoint paths from **s** to **t**.

Reduction

Consider **G** as a flow network with edge capacities **1**, and compute max-flow.

Correctness of Reduction

Lemma

If G has k edge disjoint paths $P_1, P_2, \ldots, P_k$ then there is an s-t flow of value k in G.

Proof.

Set $f(e) = 1$ if e belongs to one of the paths $P_1, P_2, \ldots, P_k$; other-wise set $f(e) = 0$. This defines a flow of value k. □

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If **G** *has a flow of value* **k** *then there are* **k** *edge disjoint paths between* **s** *and* **t**.

Proof.

1. Capacities are all **1** and hence there is integer flow of value **k**, that is $f(e) = 0$ or $f(e) = 1$ for each **e**.

2. Decompose flow into paths.

3. Flow on each path is either **1** or **0**.

4. Hence there are **k** paths $P_1, P_2, \ldots, P_k$ with flow of **1** each.

5. Paths are edge-disjoint since capacities are **1**. □

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Running Time

Theorem

*The number of edge disjoint paths in **G** can be found in **O(mn)** time.*

Proof.

1. Set capacities of edges in **G** to **1**.
2. Run Ford-Fulkerson algorithm.
3. Maximum value of flow is **n** and hence run-time is **O(nm)**.
4. Decompose flow into **k** paths (**k $\leq$ n**).
   Takes **O(k $\times$ m) = O(km) = O(mn)** time. $\qquad\qquad\square$

Remark

The algorithm also computes a set of edge-disjoint paths realizing
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Menger's Theorem

Theorem (**Menger [1927]**)

*Let **G** be a directed graph. The minimum number of edges whose removal disconnects **s** from **t** (the minimum-cut between **s** and **t**) is equal to the maximum number of edge-disjoint paths in **G** between **s** and **t**.*

Proof.

Maxflow-mincut theorem and integrality of flow.

Menger proved his theorem before Maxflow-Mincut theorem!
Maxflow-Mincut theorem is a generalization of Menger's theorem to capacitated graphs.

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Edge Disjoint Paths in Undirected Graphs

Problem

Given an undirected graph **G**, find the maximum number of edge disjoint paths in **G**

Reduction:

1. create directed graph **H** by adding directed edges (u, v) and (v, u) for each edge **uv** in **G**.

2. compute maximum **s**-**t** flow in **H**.

Problem: Both edges (u, v) and (v, u) may have non-zero flow!

Not a Problem! Can assume maximum flow in **H** is acyclic and hence cannot have non-zero flow on both (u, v) and (v, u). Reduction works. See book for more details.

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Multiple Sources and Sinks

Input:

1. A directed graph **G** with edge capacities **c(e)**.
2. Source nodes $s_1, s_2, \ldots, s_k$.
3. Sink nodes $t_1, t_2, \ldots, t_\ell$.
4. Sources and sinks are *disjoint*.

Maximum Flow: Send as much flow as possible from the sources to the sinks. *Sinks don't care which source they get flow from.*

Minimum Cut: Find a minimum capacity set of edge **E′** such that removing **E′** disconnects every source from every sink.

Multiple Sources and Sinks

Input:

1. A directed graph **G** with edge capacities **c(e)**.
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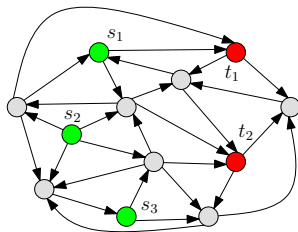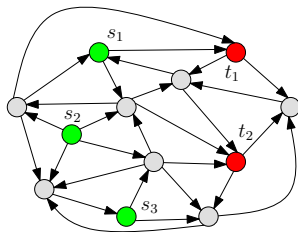4. Sources and sinks are *disjoint*.

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Multiple Sources and Sinks: Formal Definition

Input:

1. A directed graph G with edge capacities $c(e)$.
2. Source nodes $s_1, s_2, \ldots, s_k$.
3. Sink nodes $t_1, t_2, \ldots, t_\ell$.
4. Sources and sinks are *disjoint*.

A function $f : E \to \mathbb{R}^{\geq 0}$ is a **flow** if:

1. For each $e \in E$, $f(e) \leq c(e)$, and
2. for each v which is not a source or a sink $f^{in}(v) = f^{out}(v)$.

Goal: max $\sum_{i=1}^{k}(f^{out}(s_i) - f^{in}(s_i))$, that is, flow out of sources.

Reduction to Single-Source Single-Sink

1. Add a **source** node **s** and a **sink** node **t**.
2. Add edges $(s, s_1), (s, s_2), \ldots, (s, s_k)$.
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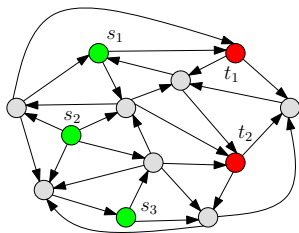
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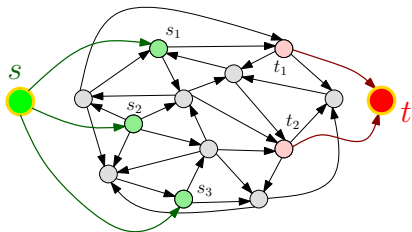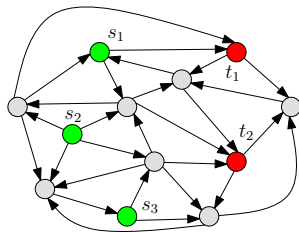
Supplies and Demands

A further generalization:

1. source s_i has a supply of $S_i \geq 0$
2. since t_j has a demand of $D_j \geq 0$ units

Question: is there a flow from source to sinks such that supplies are not exceeded and demands are met? Formally we have the additional constraints that $f^{out}(s_i) - f^{in}(s_i) \leq S_i$ for each source s_i and $f^{in}(t_j) - f^{out}(t_j) \geq D_j$ for each sink t_j.
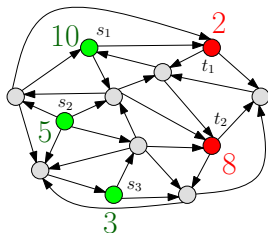
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Matching

Problem (Matching)

Input: *Given a (undirected) graph* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$.
Goal: *Find a matching of maximum cardinality.*

① *A matching is* $\mathbf{M} \subseteq \mathbf{E}$ *such that at most one edge in* $\mathbf{M}$ *is incident on any vertex*
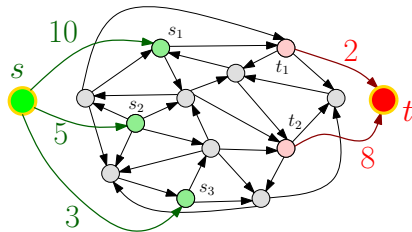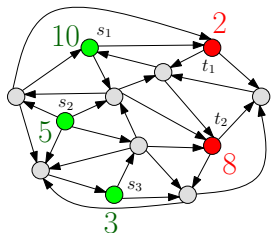
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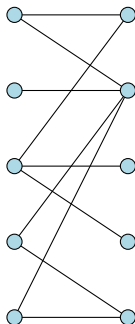
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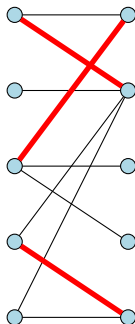
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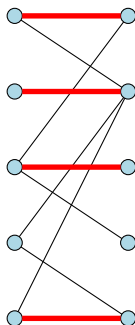
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Reduction of bipartite matching to max-flow

Max-Flow Construction

Given graph $G = (L \cup R, E)$ create flow-network $G' = (V', E')$ as follows:

1. $V' = L \cup R \cup \{s, t\}$ where s and t are the new source and sink.
2. Direct all edges in E from L to R, and add edges from s to all vertices in L and from each vertex in R to t.
3. Capacity of every edge is 1.

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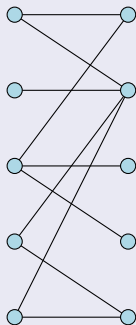
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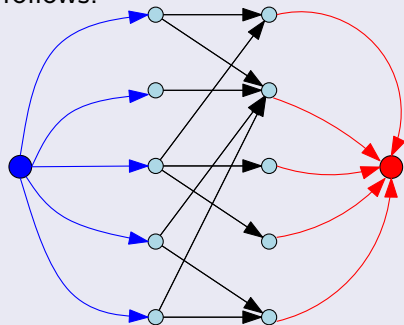

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Correctness: Matching to Flow

Proposition

If **G** has a matching of size **k** then **G'** has a flow of value **k**.

Proof.

Let **M** be matching of size **k**. Let $M = \{(u_1, v_1), \ldots, (u_k, v_k)\}$.
Consider following flow **f** in **G'**:

1. $f(s, u_i) = 1$ and $f(v_i, t) = 1$ for $1 \leq i \leq k$
2. $f(u_i, v_i) = 1$ for $1 \leq i \leq k$
3. for all other edges flow is zero.

Verify that **f** is a flow of value **k** (because **M** is a matching). □

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If G has a matching of size k then G' has a flow of value k.

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Verify that f is a flow of value k (because M is a matching). $\qquad\square$

Correctness: Flow to Matching

Proposition

If G' has a flow of value k then G has a matching of size k.

Proof.

Consider flow f of value k.

1. Can assume f is integral. Thus each edge has flow 1 or 0.
2. Consider the set M of edges from L to R that have flow 1.
   1. M has k edges because value of flow is equal to the number of non-zero flow edges crossing cut $(L \cup \{s\}, R \cup \{t\})$
   2. Each vertex has at most one edge in M incident upon it. Why?

$\square$

Correctness of Reduction

Theorem

The maximum flow value in $\mathbf{G'}$ = maximum cardinality of matching in $\mathbf{G}$.

Consequence

Thus, to find maximum cardinality matching in $\mathbf{G}$, we construct $\mathbf{G'}$ and find the maximum flow in $\mathbf{G'}$. Note that the matching itself (not just the value) can be found efficiently from the flow.

Running Time

For graph **G** with **n** vertices and **m** edges **G′** has $O(n + m)$ edges, and $O(n)$ vertices.

1. Generic Ford-Fulkerson: Running time is $O(mC) = O(nm)$ since $C = n$.
2. Capacity scaling: Running time is $O(m^2 \log C) = O(m^2 \log n)$.

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Perfect Matchings

Definition

A matching **M** is said to be **perfect** if every vertex has one edge in **M** incident upon it.
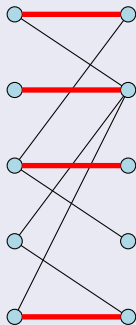
Figure: This graph does not have a perfect matching

Characterizing Perfect Matchings

Problem

When does a bipartite graph have a perfect matching?

1. Clearly $|\mathbf{L}| = |\mathbf{R}|$
2. Are there any necessary and sufficient conditions?

A Necessary Condition

Lemma

If $G = (L \cup R, E)$ has a perfect matching then for any $X \subseteq L$, $|N(X)| \geq |X|$, where $N(X)$ is the set of neighbors of vertices in X.

Proof.

Since G has a perfect matching, every vertex of X is matched to a different neighbor, and so $|N(X)| \geq |X|$. $\square$

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Hall's Theorem

Theorem (Frobenius-Hall)

Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. G has a perfect matching if and only if for every $X \subseteq L$, $|N(X)| \geq |X|$.

One direction is the necessary condition.

For the other direction we will show the following:

1. Create flow network G' from G.
2. If $|N(X)| \geq |X|$ for all X, show that minimum s-t cut in G' is of capacity $n = |L| = |R|$.
3. Implies that G has a perfect matching.

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Proof of Sufficiency

Assume $|N(X)| \geq |X|$ for any $X \subseteq L$. Then show that min **s-t** cut in G' is of capacity at least n.

Let (A, B) be an *arbitrary* **s-t** cut in G'

1. Let $X = A \cap L$ and $Y = A \cap R$.
2. Cut capacity is at least $(|L| - |X|) + |Y| + |N(X) \setminus Y|$

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Because there are...

1. $|L| - |X|$ edges from s to $L \cap B$.

2. $|Y|$ edges from Y to t.

3. there are at least $|N(X) \setminus Y|$ edges from X to vertices on the right side that are not in Y.

1. By the above, cut capacity is at least
$$\alpha = (|L| - |X|) + |Y| + |N(X) \setminus Y|.$$

2. $|N(X) \setminus Y| \geq |N(X)| - |Y|.$
   (This holds for any two sets.)

3. By assumption $|N(X)| \geq |X|$ and hence
$$|N(X) \setminus Y| \geq |N(X)| - |Y| \geq |X| - |Y|.$$

4. Cut capacity is therefore at least
$$\alpha = (|L| - |X|) + |Y| + |N(X) \setminus Y|$$
$$\geq |L| - |X| + |Y| + |X| - |Y| \geq |L| = n.$$

5. Any **s-t** cut capacity is at least **n** $\implies$ max flow at least **n** units $\implies$ perfect matching. **QED**

Hall's Theorem: Generalization

Theorem (Frobenius-Hall)

Let $G = (L \cup R, E)$ be a bipartite graph with $|L| \leq |R|$. G has a matching that matches all nodes in L if and only if for every $X \subseteq L$, $|N(X)| \geq |X|$.

Proof is essentially the same as the previous one.

1. **n** jobs, **n/2** people
2. For each job: a set of people who can do that job.
3. Each person **j** has to do exactly two jobs.
4. Goal: find an assignment of 2 jobs to each person, such that all jobs are assigned.

Solution: Build bipartite graph, compute maximum matching, remove it, compute another maximum matching. Both matchings together form a valid solution if it exists. This algorithm is

   (A) Correct.
   (B) Incorrect.

Application: Assigning jobs to people

1. **n** jobs or tasks
2. **m** people
3. for each job a set of people who can do that job
4. for each person **j** a limit on number of jobs k_j
5. Goal: find an assignment of jobs to people so that all jobs are assigned and no person is overloaded

Reduce to max-flow similar to matching.

Arises in many settings. Using *minimum-cost flows* can also handle the case when assigning a job **i** to person **j** costs c_{ij} and goal is assign all jobs but minimize cost of assignment.

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Reduction to Maximum Flow

1. Create directed graph $G = (V, E)$ as follows
   1. $V = \{s, t\} \cup L \cup R$: L set of n jobs, R set of m people
   2. add edges (s, i) for each job $i \in L$, capacity 1
   3. add edges (j, t) for each person $j \in R$, capacity k_j
   4. if job i can be done by person j add an edge (i, j), capacity 1
2. Compute max s-t flow. There is an assignment if and only if flow value is n.

Matchings in General Graphs

Matchings in general graphs more complicated.

There is a polynomial time algorithm to compute a maximum matching in a general graph. Best known running time is $\mathbf{O(m\sqrt{n})}$.

Notes

Notes

Menger, K. (1927). Zur allgemeinen kruventheorie. *Fund. Math.*, 10:96–115.