Chapter 16

Network Flows

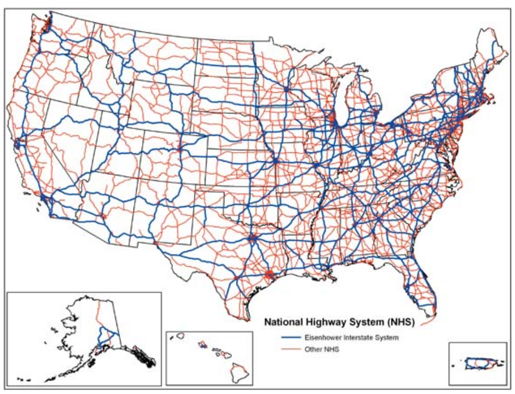
CS 473: Fundamental Algorithms, Spring 2013
March 15, 2013

16.0.0.1 Everything flows

Panta rei – everything flows (literally).
Heraclitus (535–475 BC)

16.1 Network Flows: Introduction and Setup

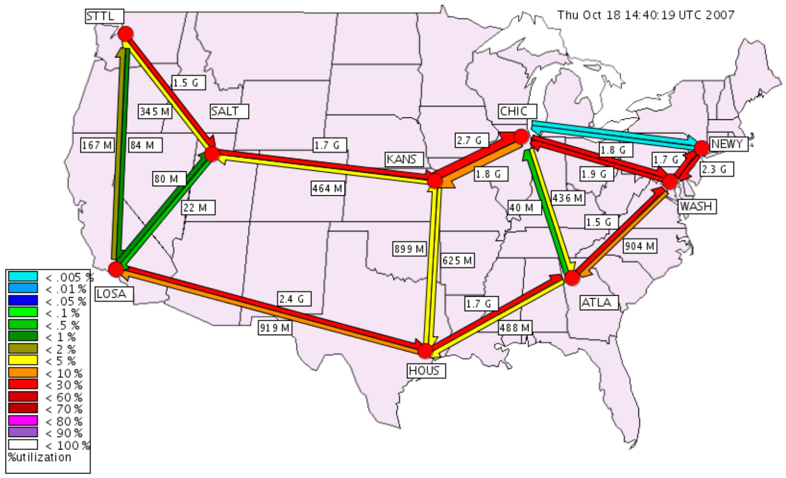
16.1.0.2 Transportation/Road Network



16.1.0.3 Internet Backbone Network

16.1.0.4 Common Features of Flow Networks

(A) ***Network*** represented by a (directed) *graph* $G = (V, E)$.
(B) Each edge e has a ***capacity*** $c(e) \geq 0$ that limits amount of *traffic* on e.
(C) *Source(s)* of traffic/data.
(D) *Sink(s)* of traffic/data.
(E) Traffic *flows* from sources to sinks.



(F) Traffic is *switched/interchanged* at nodes.

Flow abstract term to indicate stuff (traffic/data/etc) that *flows* from sources to sinks.

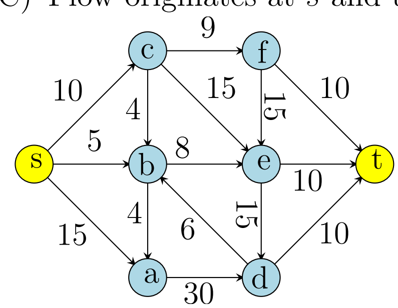
16.1.0.5 Single Source/Single Sink Flows

Simple setting:

(A) Single source s and single sink t.

(B) Every other node v is an ***internal*** node.

(C) Flow originates at s and terminates at t.



(A) Each edge e has a capacity $c(e) \geq 0$.

(B) Sometimes assume:
Source $s \in V$ has no incoming edges, and sink $t \in V$ has no outgoing edges.

Assumptions: All capacities are integer, and every vertex has at least one edge incident to it.

16.1.0.6 Definition of Flow

Two ways to define flows:

(A) edge based, or

(B) path based.

Essentially equivalent but have different uses.

Edge based definition is more compact.

16.1.0.7 Edge Based Definition of Flow

Definition 16.1.1. **Flow** *in network* $G = (V, E)$*, is function* $f : E \to \mathbb{R}^{\geq 0}$ *s.t.*

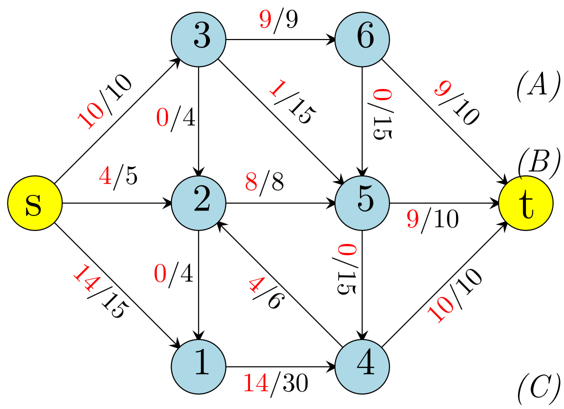


Figure 16.1: Flow with value.

(A) ***Capacity Constraint:*** *For each edge e, $f(e) \le c(e)$.*

(B) ***Conservation Constraint:*** *For each vertex $v \neq s, t$.*

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

(C) ***Value of flow****= (total flow out of source) − (total flow in to source).*

16.1.0.8 Flow...

Conservation of flow law is also known as ***Kirchhoff's law***.

16.1.0.9 More Definitions and Notation

Notation

(A) The inflow into a vertex v is $f^{\text{in}}(v) = \sum_{e \text{ into } v} f(e)$ and the outflow is $f^{\text{out}}(v) = \sum_{e \text{ out of } v} f(e)$

(B) For a set of vertices A, $f^{\text{in}}(A) = \sum_{e \text{ into } A} f(e)$. Outflow $f^{\text{out}}(A)$ is defined analogously

Definition 16.1.2. *For a network $G = (V, E)$ with source s, the* **value** *of flow f is defined as $v(f) = f^{\text{out}}(s) - f^{\text{in}}(s)$.*

16.1.0.10 A Path Based Definition of Flow

Intuition: Flow goes from source s to sink t along a path.

$\mathcal{P}$: set of all paths from s to t. $|\mathcal{P}|$ can be *exponential* in n.

Definition 16.1.3 (Flow by paths.). *A* **flow** *in network $G = (V, E)$, is function $f : \mathcal{P} \to \mathbb{R}^{\ge 0}$ s.t.*

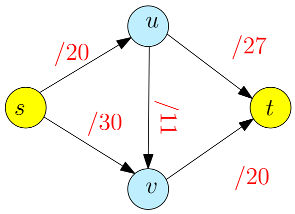
(A) ***Capacity Constraint:*** *For each edge e, total flow on e is $\le c(e)$.*

$$\sum_{p \in \mathcal{P} : e \in p} f(p) \le c(e)$$

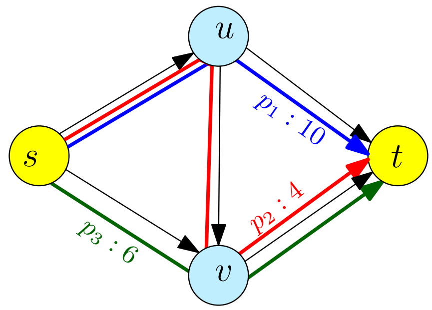
(B) ***Conservation Constraint:*** *No need! Automatic.*

Value of flow: $\sum_{p \in \mathcal{P}} f(p)$.

16.1.0.11 Example



$\mathcal{P} = \{p_1, p_2, p_3\}$
$p_1 : s \to u \to t$
$p_2 : s \to u \to v \to t$
$p_3 : s \to v \to t$
$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$



16.1.0.12 Path based flow implies edge based flow

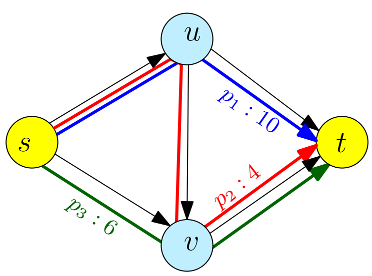
Lemma 16.1.4. *Given a path based flow $f : \mathcal{P} \to \mathbb{R}^{\geq 0}$ there is an edge based flow $f' : E \to \mathbb{R}^{\geq 0}$ of the same value.*

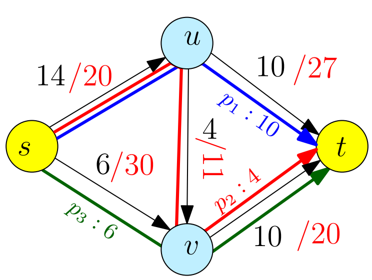
Proof: For each edge e define $f'(e) = \sum_{p: e \in p} f(p)$.

Exercise: Verify capacity and conservation constraints for f'.

Exercise: Verify that value of f and f' are equal ∎

16.1.0.13 Example





$\mathcal{P} = \{p_1, p_2, p_3\}$
$p_1 : s \to u \to t$
$p_2 : s \to u \to v \to t$
$p_3 : s \to v \to t$
$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$
$f'(s \to u) = 14$
$f'(u \to v) = 4$
$f'(s \to v) = 6$
$f'(u \to t) = 10$
$f'(v \to t) = 10$

16.1.1 Flow Decomposition

16.1.1.1 Edge based flow to Path based Flow

Lemma 16.1.5. *Given an edge based flow $f' : E \to \mathbb{R}^{\geq 0}$, there is a path based flow $f : \mathcal{P} \to \mathbb{R}^{\geq 0}$ of same value. Moreover, f assigns non-negative flow to at most m paths where $|E| = m$ and $|V| = n$. Given f', the path based flow can be computed in $O(mn)$ time.*

16.1.2 Flow Decomposition

16.1.2.1 Edge based flow to Path based Flow

Proof:[Proof Idea]
(A) Remove all edges with $f'(e) = 0$.
(B) Find a path p from s to t.
(C) Assign $f(p)$ to be $\min_{e \in p} f'(e)$.
(D) Reduce $f'(e)$ for all $e \in p$ by $f(p)$.
(E) Repeat until no path from s to t.
(F) In each iteration at least on edge has flow reduced to zero.
(G) Hence, at most m iterations. Can be implemented in $O(m(m + n))$ time. $O(mn)$ time requires care.

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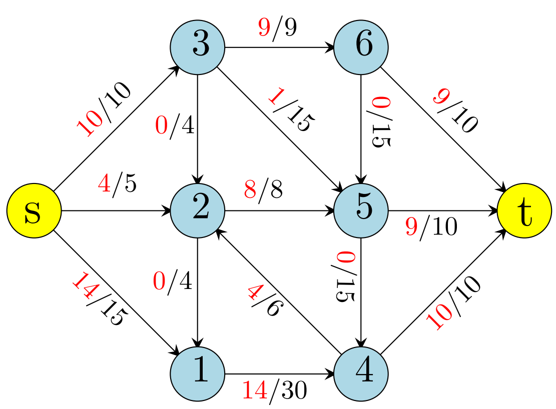
16.1.2.2 Example



16.1.2.3 Edge vs Path based Definitions of Flow

Edge based flows:
(A) *compact* representation, only m values to be specified, and
(B) need to check flow conservation explicitly at each internal node.
Path flows:
(A) in some applications, paths more natural,
(B) not compact,

(C) no need to check flow conservation constraints.
Equivalence shows that we can go back and forth easily.

16.1.2.4 The Maximum-Flow Problem

Problem

Input A network G with capacity c and source s and sink t.

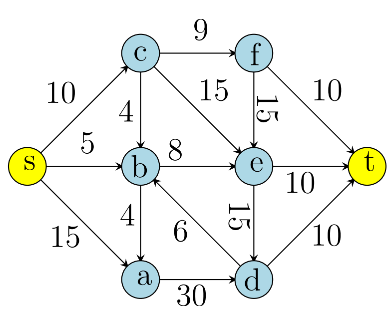
Goal Find flow of *maximum* value.

Question: Given a flow network, what is an *upper bound* on the maximum flow between source and sink?

16.1.2.5 Cuts

Definition 16.1.6 (s-t cut). *Given a flow network an* **s-t cut** *is a set of edges $E' \subset E$ such that removing E'* disconnects *s from t: in other words there is no directed $s \to t$ path in $E - E'$.*

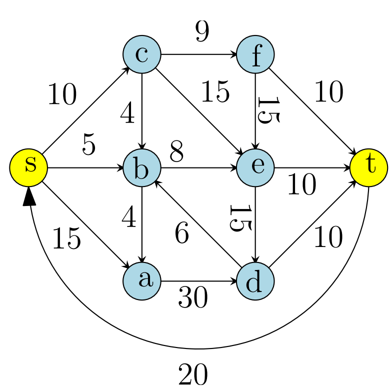
The **capacity** *of a cut E' is $c(E') = \sum_{e \in E'} c(e)$.*

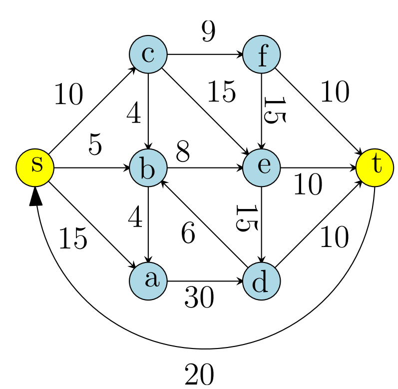


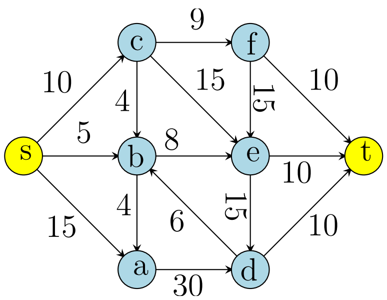
Caution:
(A) Cut may leave $t \to s$ paths!
(B) There might be many s-t cuts.

16.1.3 $s - t$ cuts

16.1.3.1 A death by a thousand cuts







16.1.3.2 Minimal Cut

Definition 16.1.7 (Minimal *s*-*t* cut.). *Given a s-t flow network $G = (V, E)$, $E' \subseteq E$ is a* **minimal cut** *if for all $e \in E'$, if $E' \setminus \{e\}$ is not a cut.*

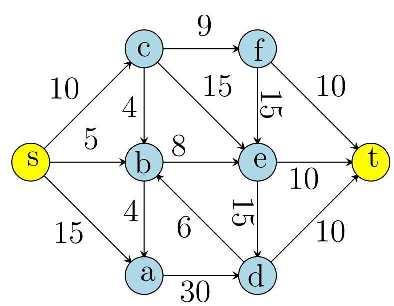
Observation: given a cut E', can check efficiently whether E' is a minimal cut or not. How?

16.1.3.3 Cuts as Vertex Partitions

Let $A \subset V$ such that
(A) $s \in A$, $t \notin A$, and
(B) $B = V \setminus A$ (hence $t \in B$).
The ***cut*** (A, B) is the set of edges

$$(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}.$$

Cut (A, B) is set of edges leaving A.



Claim 16.1.8. *(A, B) is an s-t cut.*

Proof: Let P be any $s \to t$ path in G. Since t is not in A, P has to leave A via some edge (u, v) in (A, B). ∎

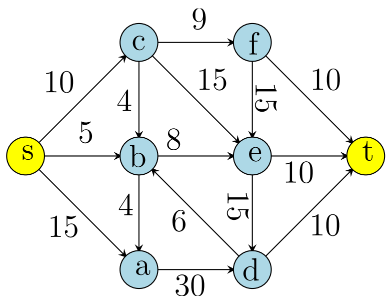
16.1.3.4 Cuts as Vertex Partitions

Lemma 16.1.9. *Suppose E' is an s-t cut. Then there is a cut (A, B) such that $(A, B) \subseteq E'$.*

Proof: E' is an s-t cut implies no path from s to t in $(V, E - E')$.
(A) Let A be set of all nodes reachable by s in $(V, E - E')$.
(B) Since E' is a cut, $t \notin A$.
(C) $(A, B) \subseteq E'$. Why?If some edge $(u, v) \in (A, B)$ is not in E' then v will be reachable by s and should be in A, hence a contradiction.

∎

Corollary 16.1.10. *Every* minimal *s-t cut E' is a cut of the form (A, B).*



16.1.3.5 Minimum Cut

Definition 16.1.11. *Given a flow network an s-t* **minimum** *cut is a cut E' of smallest capacity amongst all s-t cuts.*

Observation: exponential number of s-t cuts and no "easy" algorithm to find a minimum cut.

16.1.3.6 The Minimum-Cut Problem

Problem

Input A flow network G

Goal Find the capacity of a *minimum* s-t cut

16.1.3.7 Flows and Cuts

Lemma 16.1.12. *For any s-t cut E',* **maximum** *s-t flow $\leq$ capacity of E'.*

Proof: Formal proof easier with path based definition of flow.

Suppose $f : \mathcal{P} \to \mathbb{R}^{\geq 0}$ is a max-flow. Every path $p \in \mathcal{P}$ contains an edge $e \in E'$. Why? Assign each path $p \in \mathcal{P}$ to exactly one edge $e \in E'$.

Let $\mathcal{P}_e$ be paths assigned to $e \in E'$. Then

$$v(f) = \sum_{p \in \mathcal{P}} f(p) = \sum_{e \in E'} \sum_{p \in \mathcal{P}_e} f(p) \leq \sum_{e \in E'} c(e).$$

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16.1.3.8 Flows and Cuts

Lemma 16.1.13. *For any s-t cut E',* **maximum** *s-t flow $\leq$ capacity of E'.*

Corollary 16.1.14. *Maximum s-t flow $\leq$ minimum s-t cut.*

16.1.3.9 Max-Flow Min-Cut Theorem

Theorem 16.1.15. *In any flow network the maximum s-t flow is* equal *to the minimum s-t cut.*

Can compute minimum-cut from maximum flow and vice-versa!
Proof coming shortly.
Many applications:

(A) optimization
(B) graph theory
(C) combinatorics

16.1.3.10 The Maximum-Flow Problem

Problem

Input A network G with capacity c and source s and sink t.

Goal Find flow of *maximum* value from s to t.

Exercise: Given G, s, t as above, show that one can remove all edges into s and all edges out of t without affecting the flow value between s and t.