# Network Flows

Lecture 16

March 15, 2013

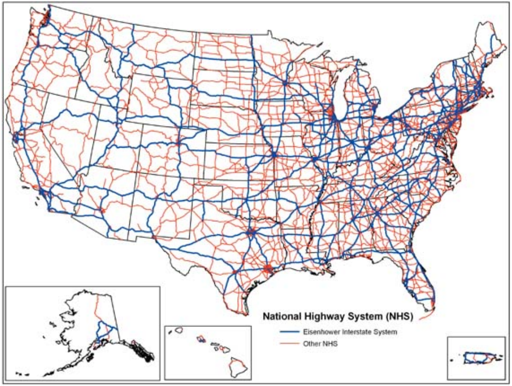
# Everything flows

**Panta rei** – everything flows (literally).
Heraclitus (535–475 BC)

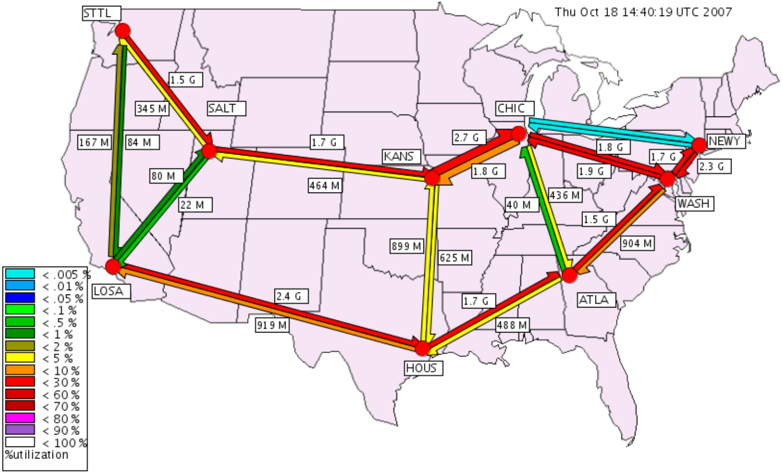
# Part I

# Network Flows: Introduction and Setup

# Transportation/Road Network



# Internet Backbone Network



# Common Features of Flow Networks

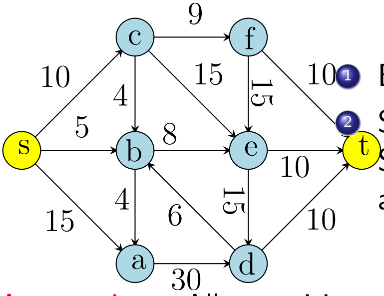
1. **Network** represented by a (directed) *graph* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$.
2. Each edge $\mathbf{e}$ has a **capacity** $\mathbf{c(e)} \geq \mathbf{0}$ that limits amount of *traffic* on $\mathbf{e}$.
3. *Source(s)* of traffic/data.
4. *Sink(s)* of traffic/data.
5. Traffic *flows* from sources to sinks.
6. Traffic is *switched/interchanged* at nodes.

**Flow** abstract term to indicate stuff (traffic/data/etc) that **flows** from sources to sinks.

# Single Source/Single Sink Flows

Simple setting:

1. Single source **s** and single sink **t**.
2. Every other node **v** is an **internal** node.
3. Flow originates at **s** and terminates at **t**.




1. Each edge **e** has a capacity $\mathbf{c(e)} \geq \mathbf{0}$.
2. Sometimes assume:
   Source $\mathbf{s} \in \mathbf{V}$ has no incoming edges, and sink $\mathbf{t} \in \mathbf{V}$ has no outgoing edges.

Assumptions: All capacities are integer, and every vertex has at least one edge incident to it.

# Definition of Flow

Two ways to define flows:

1. edge based, or
2. path based.

Essentially equivalent but have different uses.

Edge based definition is more compact.

# Edge Based Definition of Flow

## Definition

**Flow** in network $G = (V, E)$, is function $\mathbf{f : E \to \mathbb{R}^{\geq 0}}$ s.t.

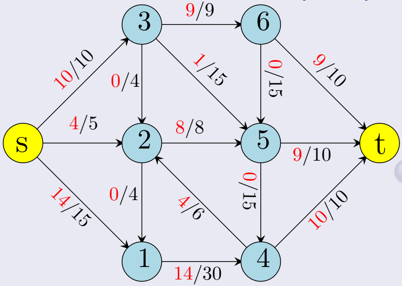



Figure: Flow with value.

1. Capacity Constraint: For each edge $e$, $f(e) \leq c(e)$.

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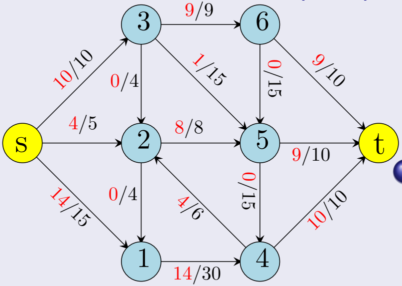


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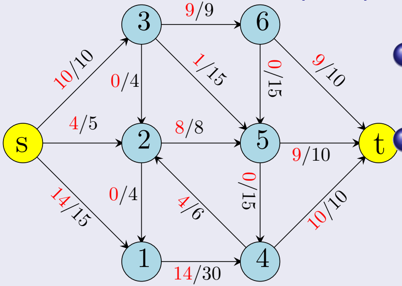


Figure: Flow with value.

1. Capacity Constraint: For each edge $\mathbf{e}$, $\mathbf{f(e)} \leq \mathbf{c(e)}$.
2. Conservation Constraint: For each vertex $\mathbf{v} \neq \mathbf{s}, \mathbf{t}$.

$$\sum_{\mathbf{e} \text{ into } \mathbf{v}} \mathbf{f(e)} = \sum_{\mathbf{e} \text{ out of } \mathbf{v}} \mathbf{f(e)}$$

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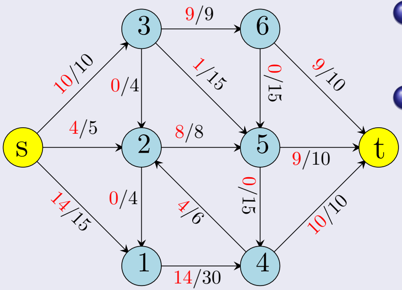


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3. Value of flow= (total flow out of source) − (total flow in to source).

# Flow...

Conservation of flow law is also known as **Kirchhoff's law**.

# More Definitions and Notation

## Notation

1. The inflow into a vertex $\mathbf{v}$ is $\mathbf{f}^{\mathrm{in}}(\mathbf{v}) = \sum_{\mathbf{e} \text{ into } \mathbf{v}} \mathbf{f}(\mathbf{e})$ and the outflow is $\mathbf{f}^{\mathrm{out}}(\mathbf{v}) = \sum_{\mathbf{e} \text{ out of } \mathbf{v}} \mathbf{f}(\mathbf{e})$
2. For a set of vertices $\mathbf{A}$, $\mathbf{f}^{\mathrm{in}}(\mathbf{A}) = \sum_{\mathbf{e} \text{ into } \mathbf{A}} \mathbf{f}(\mathbf{e})$. Outflow $\mathbf{f}^{\mathrm{out}}(\mathbf{A})$ is defined analogously

## Definition

For a network $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with source $\mathbf{s}$, the **value** of flow $\mathbf{f}$ is defined as $\mathbf{v}(\mathbf{f}) = \mathbf{f}^{\mathrm{out}}(\mathbf{s}) - \mathbf{f}^{\mathrm{in}}(\mathbf{s})$.

# A Path Based Definition of Flow

Intuition: Flow goes from source **s** to sink **t** along a path.

$\mathcal{P}$: set of all paths from **s** to **t**. $|\mathcal{P}|$ can be **exponential** in **n**.

## Definition (Flow by paths.)

A **flow** in network $G = (V, E)$, is function $\mathbf{f} : \mathcal{P} \to \mathbb{R}^{\geq 0}$ s.t.

1. Capacity Constraint: For each edge **e**, total flow on **e** is $\leq \mathbf{c(e)}$.

$$\sum_{p \in \mathcal{P} : e \in p} \mathbf{f(p)} \leq \mathbf{c(e)}$$

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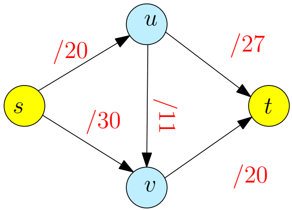
# Example




$\mathcal{P} = \{p_1, p_2, p_3\}$
$p_1 : s \to u \to t$
$p_2 : s \to u \to v \to t$
$p_3 : s \to v \to t$

$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$

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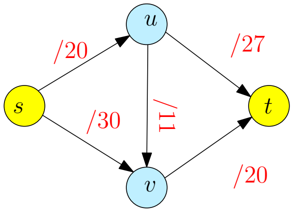


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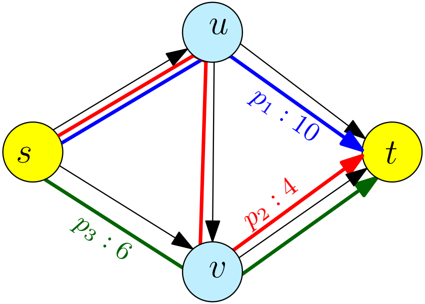

# Path based flow implies edge based flow

## Lemma

*Given a path based flow* $\mathbf{f} : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ *there is an edge based flow* $\mathbf{f'} : \mathbf{E} \rightarrow \mathbb{R}^{\geq 0}$ *of the same value.*

## Proof.

For each edge $\mathbf{e}$ define $\mathbf{f'(e)} = \sum_{\mathbf{p:e \in p}} \mathbf{f(p)}$.

**Exercise:** Verify capacity and conservation constraints for $\mathbf{f'}$.

**Exercise:** Verify that value of $\mathbf{f}$ and $\mathbf{f'}$ are equal □

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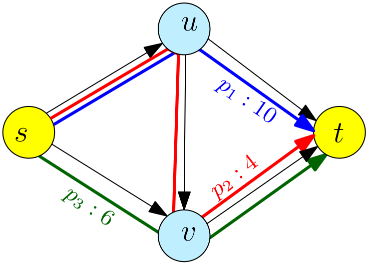


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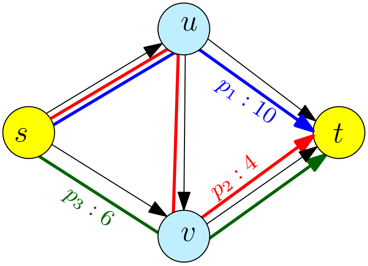


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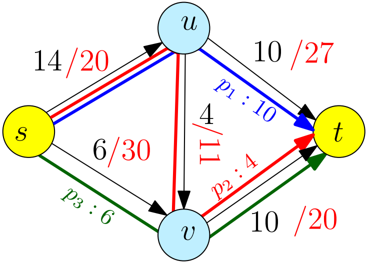


$f'(s \to u) = 14$
$f'(u \to v) = 4$
$f'(s \to v) = 6$
$f'(u \to t) = 10$
$f'(v \to t) = 10$

# Flow Decomposition

## Edge based flow to Path based Flow

### Lemma

*Given an edge based flow* $\mathbf{f}' : \mathbf{E} \rightarrow \mathbb{R}^{\geq 0}$, *there is a path based flow* $\mathbf{f} : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ *of same value. Moreover,* $\mathbf{f}$ *assigns non-negative flow to at most* $\mathbf{m}$ *paths where* $|\mathbf{E}| = \mathbf{m}$ *and* $|\mathbf{V}| = \mathbf{n}$. *Given* $\mathbf{f}'$, *the path based flow can be computed in* $\mathbf{O(mn)}$ *time.*

# Flow Decomposition

## Edge based flow to Path based Flow

### Proof Idea.

1. Remove all edges with $\mathbf{f'(e) = 0}$.
2. Find a path $\mathbf{p}$ from $\mathbf{s}$ to $\mathbf{t}$.
3. Assign $\mathbf{f(p)}$ to be $\mathbf{\min_{e \in p} f'(e)}$.
4. Reduce $\mathbf{f'(e)}$ for all $\mathbf{e \in p}$ by $\mathbf{f(p)}$.
5. Repeat until no path from $\mathbf{s}$ to $\mathbf{t}$.
6. In each iteration at least on edge has flow reduced to zero.
7. Hence, at most $\mathbf{m}$ iterations. Can be implemented in $\mathbf{O(m(m+n))}$ time. $\mathbf{O(mn)}$ time requires care.

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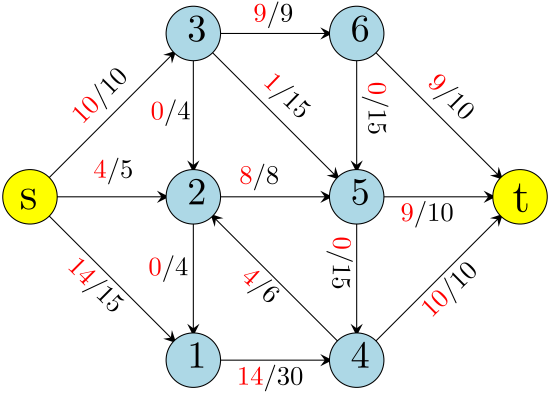

s
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# Edge vs Path based Definitions of Flow

Edge based flows:

1. **compact** representation, only **m** values to be specified, and
2. need to check flow conservation explicitly at each internal node.

Path flows:

1. in some applications, paths more natural,
2. not compact,
3. no need to check flow conservation constraints.

Equivalence shows that we can go back and forth easily.

# The Maximum-Flow Problem

## Problem

**Input** A network $\mathbf{G}$ with capacity $\mathbf{c}$ and source $\mathbf{s}$ and sink $\mathbf{t}$.

**Goal** Find flow of **maximum** value.

Question: Given a flow network, what is an *upper bound* on the maximum flow between source and sink?

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# Cuts

## Definition (s-t cut)

Given a flow network an **s-t cut** is a set of edges $\mathbf{E}' \subset \mathbf{E}$ such that removing $\mathbf{E}'$ *disconnects* $\mathbf{s}$ from $\mathbf{t}$: in other words there is no directed $\mathbf{s} \to \mathbf{t}$ path in $\mathbf{E} - \mathbf{E}'$.
The **capacity** of a cut $\mathbf{E}'$ is $\mathbf{c}(\mathbf{E}') = \sum_{\mathbf{e} \in \mathbf{E}'} \mathbf{c}(\mathbf{e})$.

Caution:

1. Cut may leave $\mathbf{t} \to \mathbf{s}$ paths!
2. There might be many $\mathbf{s}$-$\mathbf{t}$ cuts.

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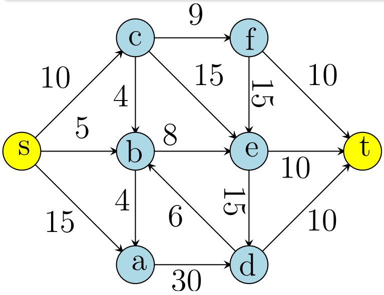


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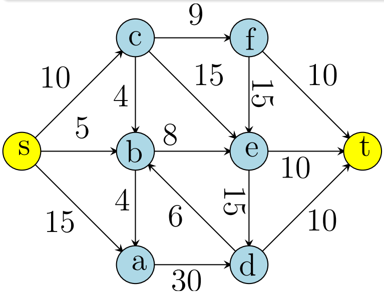


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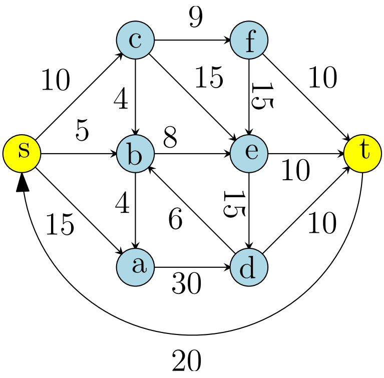
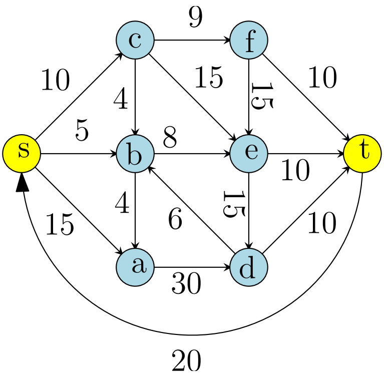
# s — t cuts

## A death by a thousand cuts

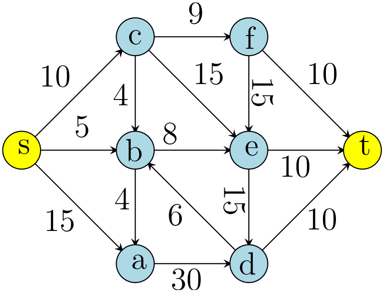


# Minimal Cut

## Definition (Minimal s-t cut.)

Given a **s**-**t** flow network $G = (V, E)$, $E' \subseteq E$ is a **minimal cut** if for all $\mathbf{e} \in E'$, if $E' \setminus \{\mathbf{e}\}$ is not a cut.

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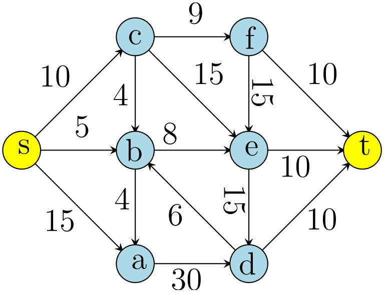


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# Cuts as Vertex Partitions

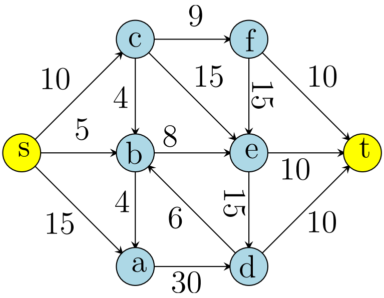
Let $\mathbf{A} \subset \mathbf{V}$ such that

1. $\mathbf{s} \in \mathbf{A}$, $\mathbf{t} \notin \mathbf{A}$, and
2. $\mathbf{B} = \mathbf{V} \setminus \mathbf{A}$ (hence $\mathbf{t} \in \mathbf{B}$).

The **cut** $(\mathbf{A}, \mathbf{B})$ is the set of edges

$$(\mathbf{A}, \mathbf{B}) = \{(\mathbf{u}, \mathbf{v}) \in \mathbf{E} \mid \mathbf{u} \in \mathbf{A}, \mathbf{v} \in \mathbf{B}\}.$$

Cut $(\mathbf{A}, \mathbf{B})$ is set of edges leaving $\mathbf{A}$.




## Claim

$(\mathbf{A}, \mathbf{B})$ *is an* $\mathbf{s}$-$\mathbf{t}$ *cut.*

## Proof.

Let $\mathbf{P}$ be any $\mathbf{s} \to \mathbf{t}$ path in $\mathbf{G}$. Since $\mathbf{t}$ is not in $\mathbf{A}$, $\mathbf{P}$ has to leave $\mathbf{A}$ via some edge $(\mathbf{u}, \mathbf{v})$ in $(\mathbf{A}, \mathbf{B})$. $\square$

# Cuts as Vertex Partitions

**Lemma**

*Suppose* $\mathbf{E'}$ *is an* $\mathbf{s}$-$\mathbf{t}$ *cut. Then there is a cut* $(\mathbf{A}, \mathbf{B})$ *such that* $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{E'}$.

**Proof.**

$\mathbf{E'}$ is an $\mathbf{s}$-$\mathbf{t}$ cut implies no path from $\mathbf{s}$ to $\mathbf{t}$ in $(\mathbf{V}, \mathbf{E} - \mathbf{E'})$.

1. Let $\mathbf{A}$ be set of all nodes reachable by $\mathbf{s}$ in $(\mathbf{V}, \mathbf{E} - \mathbf{E'})$.
2. Since $\mathbf{E'}$ is a cut, $\mathbf{t} \not\in \mathbf{A}$.
3. $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{E'}$. Why?

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**Corollary**

*Every* minimal $\mathbf{s}$-$\mathbf{t}$ *cut* $\mathbf{E'}$ *is a cut of the form* $(\mathbf{A}, \mathbf{B})$.

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$\mathbf{E'}$ is an $\mathbf{s}$-$\mathbf{t}$ cut implies no path from $\mathbf{s}$ to $\mathbf{t}$ in $(\mathbf{V}, \mathbf{E} - \mathbf{E'})$.

1. Let $\mathbf{A}$ be set of all nodes reachable by $\mathbf{s}$ in $(\mathbf{V}, \mathbf{E} - \mathbf{E'})$.
2. Since $\mathbf{E'}$ is a cut, $\mathbf{t} \not\in \mathbf{A}$.
3. $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{E'}$. Why?If some edge $(\mathbf{u}, \mathbf{v}) \in (\mathbf{A}, \mathbf{B})$ is not in $\mathbf{E'}$ then $\mathbf{v}$ will be reachable by $\mathbf{s}$ and should be in $\mathbf{A}$, hence a contradiction. □

## Corollary

*Every* minimal $\mathbf{s}$-$\mathbf{t}$ *cut* $\mathbf{E'}$ *is a cut of the form* $(\mathbf{A}, \mathbf{B})$.

# Cuts as Vertex Partitions

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*Suppose* $\mathbf{E'}$ *is an* $\mathbf{s}$-$\mathbf{t}$ *cut. Then there is a cut* $\mathbf{(A, B)}$ *such that* $\mathbf{(A, B)} \subseteq \mathbf{E'}$*.*

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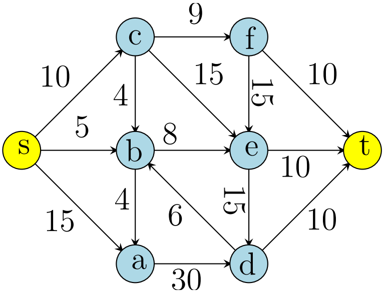
# Minimum Cut

## Definition

Given a flow network an **s-t** **minimum** cut is a cut **E′** of smallest capacity amongst all **s-t** cuts.

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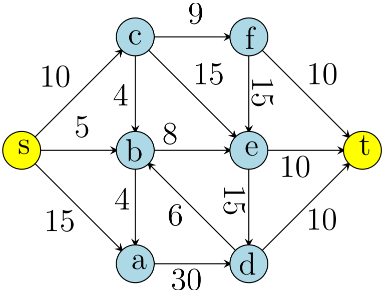


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# The Minimum-Cut Problem

## Problem

Input A flow network **G**

Goal Find the capacity of a *minimum* **s**-**t** cut

# Flows and Cuts

## Lemma

*For any* **s-t** *cut* $\mathbf{E'}$, **maximum** **s-t** *flow* $\leq$ *capacity of* $\mathbf{E'}$.

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Formal proof easier with path based definition of flow.
Suppose $\mathbf{f} : \mathcal{P} \to \mathbb{R}^{\geq 0}$ is a max-flow.

Every path $\mathbf{p} \in \mathcal{P}$ contains an edge $\mathbf{e} \in \mathbf{E'}$. Why?
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# Max-Flow Min-Cut Theorem

## Theorem

*In any flow network the maximum* **s-t** *flow is equal to the minimum* **s-t** *cut.*

Can compute minimum-cut from maximum flow and vice-versa!

Proof coming shortly.

Many applications:

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Input A network **G** with capacity **c** and source **s** and sink **t**.

Goal Find flow of **maximum** value from **s** to **t**.

**Exercise:** Given **G**, **s**, **t** as above, show that one can remove all edges into **s** and all edges out of **t** without affecting the flow value between **s** and **t**.

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