Chapter 13

Introduction to Randomized Algorithms: QuickSort and QuickSelect

CS 473: Fundamental Algorithms, Spring 2013
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13.1 Introduction to Randomized Algorithms

13.2 Introduction

13.2.0.1 Randomized Algorithms

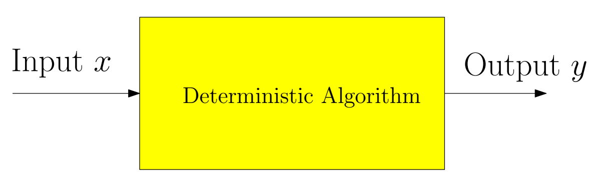
13.2.0.2 Example: Randomized QuickSort

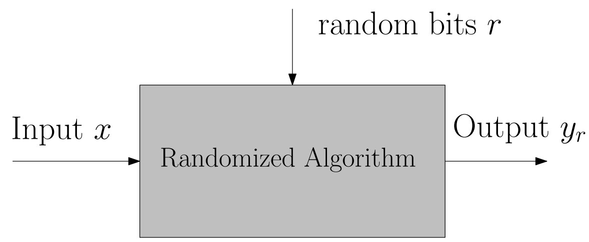
QuickSort Hoare [1962]

(A) Pick a pivot element from array
(B) Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
(C) Recursively sort the subarrays, and concatenate them.

Randomized **QuickSort**

(A) Pick a pivot element *uniformly at random* from the array
(B) Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
(C) Recursively sort the subarrays, and concatenate them.





13.2.0.3 Example: Randomized Quicksort

Recall: **QuickSort** can take $\Omega(n^2)$ time to sort array of size n.

Theorem 13.2.1. *Randomized* **QuickSort** *sorts a given array of length n in $O(n \log n)$* expected *time.*

Note: On *every* input randomized **QuickSort** takes $O(n \log n)$ time in expectation. On *every* input it may take $\Omega(n^2)$ time with some small probability.

13.2.0.4 Example: Verifying Matrix Multiplication

Problem Given three $n \times n$ matrices A, B, C is $AB = C$?

Deterministic algorithm:

(A) Multiply A and B and check if equal to C.
(B) Running time? $O(n^3)$ by straight forward approach. $O(n^{2.37})$ with fast matrix multiplication (complicated and impractical).

13.2.0.5 Example: Verifying Matrix Multiplication

Problem Given three $n \times n$ matrices A, B, C is $AB = C$?

Randomized algorithm:

(A) Pick a random $n \times 1$ vector r.
(B) Return the answer of the equality $ABr = Cr$.
(C) Running time? $O(n^2)$!

Theorem 13.2.2. *If $AB = C$ then the algorithm will always say YES. If $AB \neq C$ then the algorithm will say YES with probability at most $1/2$. Can repeat the algorithm 100 times independently to reduce the probability of a false positive to $1/2^{100}$.*

13.2.0.6 Why randomized algorithms?

(A) Many many applications in algorithms, data structures and computer science!
(B) In some cases only known algorithms are randomized or randomness is provably necessary.
(C) Often randomized algorithms are (much) simpler and/or more efficient.
(D) Several deep connections to mathematics, physics etc.
(E) ...
(F) Lots of fun!

13.2.0.7 Where do I get random bits?

Question: Are true random bits available in practice?
(A) Buy them!
(B) CPUs use physical phenomena to generate random bits.
(C) Can use pseudo-random bits or semi-random bits from nature. Several fundamental unresolved questions in complexity theory on this topic. Beyond the scope of this course.
(D) In practice pseudo-random generators work quite well in many applications.
(E) The model is interesting to think in the abstract and is very useful even as a theoretical construct. One can *derandomize* randomized algorithms to obtain deterministic algorithms.

13.2.0.8 Average case analysis vs Randomized algorithms

Average case analysis:
(A) Fix a deterministic algorithm.
(B) Assume inputs comes from a probability distribution.
(C) Analyze the algorithm's *average* performance over the distribution over inputs.

Randomized algorithms:
(A) Algorithm uses random bits in addition to input.
(B) Analyze algorithms *average* performance over the given input where the average is over the random bits that the algorithm uses.
(C) On each input behaviour of algorithm is random. Analyze worst-case over all inputs of the (average) performance.

13.3 Basics of Discrete Probability

13.3.0.9 Discrete Probability

We restrict attention to finite probability spaces.

Definition 13.3.1. *A discrete probability space is a pair* $(\Omega, \mathbf{Pr})$ *consists of finite set* Ω *of* **elementary events** *and function* $p : \Omega \to [0, 1]$ *which assigns a probability* $\mathbf{Pr}[\omega]$ *for each* $\omega \in \Omega$ *such that* $\sum_{\omega\in\Omega} \mathbf{Pr}[\omega] = 1$.

Example 13.3.2. *An unbiased coin.* $\Omega = \{H, T\}$ *and* $\mathbf{Pr}[H] = \mathbf{Pr}[T] = 1/2$.

Example 13.3.3. *A* 6*-sided unbiased die.* $\Omega = \{1, 2, 3, 4, 5, 6\}$ *and* $\mathbf{Pr}[i] = 1/6$ *for* $1 \leq i \leq 6$.

13.3.1 Discrete Probability

13.3.1.1 And more examples

Example 13.3.4. *A biased coin.* $\Omega = \{H, T\}$ *and* $\mathbf{Pr}[H] = 2/3, \mathbf{Pr}[T] = 1/3$.

Example 13.3.5. *Two independent unbiased coins.* $\Omega = \{HH, TT, HT, TH\}$ *and* $\mathbf{Pr}[HH] = \mathbf{Pr}[TT] = \mathbf{Pr}[HT] = \mathbf{Pr}[TH] = 1/4$.

Example 13.3.6. *A pair of (highly) correlated dice.*
$\Omega = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$.
$\mathbf{Pr}[i, i] = 1/6$ *for* $1 \leq i \leq 6$ *and* $\mathbf{Pr}[i, j] = 0$ *if* $i \neq j$.

13.3.1.2 Events

Definition 13.3.7. *Given a probability space* $(\Omega, \mathbf{Pr})$ *an* **event** *is a subset of* Ω. *In other words an event is a collection of elementary events. The probability of an event* A, *denoted by* $\mathbf{Pr}[A]$, *is* $\sum_{\omega \in A} \mathbf{Pr}[\omega]$.

The **complement event** *of an event* $A \subseteq \Omega$ *is the event* $\Omega \setminus A$ *frequently denoted by* $\bar{A}$.

13.3.2 Events

13.3.2.1 Examples

Example 13.3.8. *A pair of independent dice.* $\Omega = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$.

(A) Let A *be the event that the sum of the two numbers on the dice is even.*
Then $A = \left\{ (i, j) \in \Omega \;\middle|\; (i + j) \text{ is even} \right\}$.
$\mathbf{Pr}[A] = |A|/36 = 1/2$.

(B) Let B *be the event that the first die has* 1. *Then* $B = \left\{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6) \right\}$.
$\mathbf{Pr}[B] = 6/36 = 1/6$.

13.3.2.2 Independent Events

Definition 13.3.9. *Given a probability space* $(\Omega, \mathbf{Pr})$ *and two events* A, B *are* **independent** *if and only if* $\mathbf{Pr}[A \cap B] = \mathbf{Pr}[A] \mathbf{Pr}[B]$. *Otherwise they are* dependent. *In other words* A, B *independent implies one does not affect the other.*

Example 13.3.10. *Two coins.* $\Omega = \{HH, TT, HT, TH\}$ *and* $\mathbf{Pr}[HH] = \mathbf{Pr}[TT] = \mathbf{Pr}[HT] = \mathbf{Pr}[TH] = 1/4$.

(A) A *is the event that the first coin is heads and* B *is the event that second coin is tails.* A, B *are independent.*

(B) A *is the event that the two coins are different.* B *is the event that the second coin is heads.* A, B *independent.*

13.3.3 Independent Events

13.3.3.1 Examples

Example 13.3.11. A *is the event that both are not tails and* B *is event that second coin is heads.* A, B *are dependent.*

13.3.4 Union bound

13.3.4.1 The probability of the union of two events, is no bigger than the probability of the sum of their probabilities.

Lemma 13.3.12. *For any two events $\mathcal{E}$ and $\mathcal{F}$, we have that* $\mathbf{Pr}\left[\mathcal{E} \cup \mathcal{F}\right] \leq \mathbf{Pr}\left[\mathcal{E}\right] + \mathbf{Pr}\left[\mathcal{F}\right]$.

Proof: Consider $\mathcal{E}$ and $\mathcal{F}$ to be a collection of elmentery events (which they are). We have

$$\begin{aligned}\mathbf{Pr}\left[\mathcal{E} \cup \mathcal{F}\right] &= \sum_{x \in \mathcal{E} \cup \mathcal{F}} \mathbf{Pr}[x] \\ &\leq \sum_{x \in \mathcal{E}} \mathbf{Pr}[x] + \sum_{x \in \mathcal{F}} \mathbf{Pr}[x] = \mathbf{Pr}\left[\mathcal{E}\right] + \mathbf{Pr}\left[\mathcal{F}\right].\end{aligned}$$

■

13.3.4.2 Random Variables

Definition 13.3.13. *Given a probability space $(\Omega, \mathbf{Pr})$ a (real-valued) random variable X over Ω is a function that maps each elementary event to a real number. In other words $X : \Omega \rightarrow \mathbb{R}$.*

Example 13.3.14. *A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbf{Pr}[i] = 1/6$ for $1 \leq i \leq 6$.*
(A) $X : \Omega \rightarrow \mathbb{R}$ where $X(i) = i \mod 2$.
(B) $Y : \Omega \rightarrow \mathbb{R}$ where $Y(i) = i^2$.

Definition 13.3.15. *A* **binary random variable** *is one that takes on values in $\{0, 1\}$.*

13.3.4.3 Indicator Random Variables

Special type of random variables that are quite useful.

Definition 13.3.16. *Given a probability space $(\Omega, \mathbf{Pr})$ and an event $A \subseteq \Omega$ the indicator random variable X_A is a binary random variable where $X_A(\omega) = 1$ if $\omega \in A$ and $X_A(\omega) = 0$ if $\omega \notin A$.*

Example 13.3.17. *A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbf{Pr}[i] = 1/6$ for $1 \leq i \leq 6$. Let A be the even that i is divisible by 3. Then $X_A(i) = 1$ if $i = 3, 6$ and 0 otherwise.*

13.3.4.4 Expectation

Definition 13.3.18. *For a random variable X over a probability space $(\Omega, \mathbf{Pr})$ the* **expectation** *of X is defined as $\sum_{\omega \in \Omega} \mathbf{Pr}[\omega]\, X(\omega)$. In other words, the expectation is the average value of X according to the probabilities given by $\mathbf{Pr}[\cdot]$.*

Example 13.3.19. *A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbf{Pr}[i] = 1/6$ for $1 \leq i \leq 6$.*
(A) $X : \Omega \rightarrow \mathbb{R}$ where $X(i) = i \mod 2$. Then $\mathbf{E}[X] = 1/2$.
(B) $Y : \Omega \rightarrow \mathbb{R}$ where $Y(i) = i^2$. Then $\mathbf{E}[Y] = \sum_{i=1}^{6} \frac{1}{6} \cdot i^2 = 91/6$.

13.3.4.5 Expectation

Proposition 13.3.20. *For an indicator variable X_A, $\mathbf{E}[X_A] = \mathbf{Pr}[A]$.*

Proof:

$$
\begin{aligned}
\mathbf{E}[X_A] &= \sum_{y \in \Omega} X_A(y)\, \mathbf{Pr}[y] \\
&= \sum_{y \in A} 1 \cdot \mathbf{Pr}[y] + \sum_{y \in \Omega \setminus A} 0 \cdot \mathbf{Pr}[y] \\
&= \sum_{y \in A} \mathbf{Pr}[y] \\
&= \mathbf{Pr}[A]\,.
\end{aligned}
$$

■

13.3.4.6 Linearity of Expectation

Lemma 13.3.21. *Let X, Y be two random variables (not necessarily independent) over a probability space $(\Omega, \mathbf{Pr})$. Then $\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$.*

Proof:

$$
\begin{aligned}
\mathbf{E}[X+Y] &= \sum_{\omega \in \Omega} \mathbf{Pr}[\omega]\, (X(\omega) + Y(\omega)) \\
&= \sum_{\omega \in \Omega} \mathbf{Pr}[\omega]\, X(\omega) + \sum_{\omega \in \Omega} \mathbf{Pr}[\omega]\, Y(\omega) = \mathbf{E}[X] + \mathbf{E}[Y]\,.
\end{aligned}
$$

■

Corollary 13.3.22. $\mathbf{E}[a_1 X_1 + a_2 X_2 + \ldots + a_n X_n] = \sum_{i=1}^{n} a_i\, \mathbf{E}[X_i]$.

13.4 Analyzing Randomized Algorithms

13.4.0.7 Types of Randomized Algorithms

Typically one encounters the following types:

(A) ***Las Vegas randomized algorithms:*** for a given input x output of algorithm is always correct but the running time is a random variable. In this case we are interested in analyzing the *expected* running time.

(B) ***Monte Carlo randomized algorithms:*** for a given input x the running time is deterministic but the output is random; correct with some probability. In this case we are interested in analyzing the *probability* of the correct output (and also the running time).

(C) Algorithms whose running time and output may both be random.

13.4.0.8 Analyzing Las Vegas Algorithms

Deterministic algorithm Q for a problem Π:

(A) Let $Q(x)$ be the time for Q to run on input x of length $|x|$.
(B) Worst-case analysis: run time on worst input for a given size n.

$$T_{wc}(n) = \max_{x:|x|=n} Q(x).$$

Randomized algorithm R for a problem Π:

(A) Let $R(x)$ be the time for Q to run on input x of length $|x|$.
(B) $R(x)$ is a random variable: depends on random bits used by R.
(C) $\mathbf{E}[R(x)]$ is the expected running time for R on x
(D) Worst-case analysis: expected time on worst input of size n

$$T_{rand-wc}(n) = \max_{x:|x|=n} \mathbf{E}[Q(x)].$$

13.4.0.9 Analyzing Monte Carlo Algorithms

Randomized algorithm M for a problem Π:

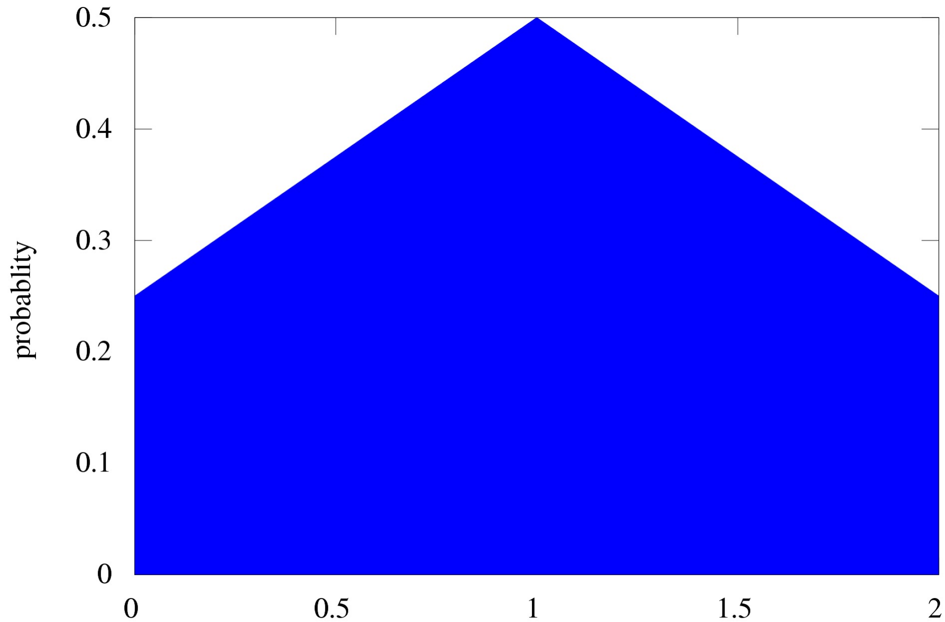
(A) Let $M(x)$ be the time for M to run on input x of length $|x|$. For Monte Carlo, assumption is that run time is deterministic.
(B) Let $\mathbf{Pr}[x]$ be the probability that M is correct on x.
(C) $\mathbf{Pr}[x]$ is a random variable: depends on random bits used by M.
(D) Worst-case analysis: success probability on worst input

$$P_{rand-wc}(n) = \min_{x:|x|=n} \mathbf{Pr}[x].$$

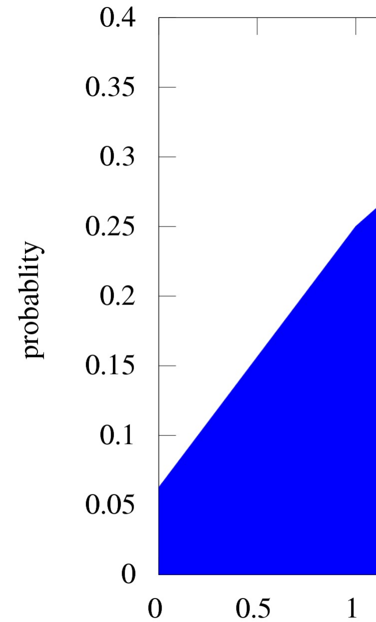
13.5 Why does randomization help?

13.5.0.10 Massive randomness.. Is not that random.

Consider flipping a fair coin n times independently, head given 1, tail gives zero. How many heads? ...we get a binomial distribution.


probablity
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2

n = 2


probablity
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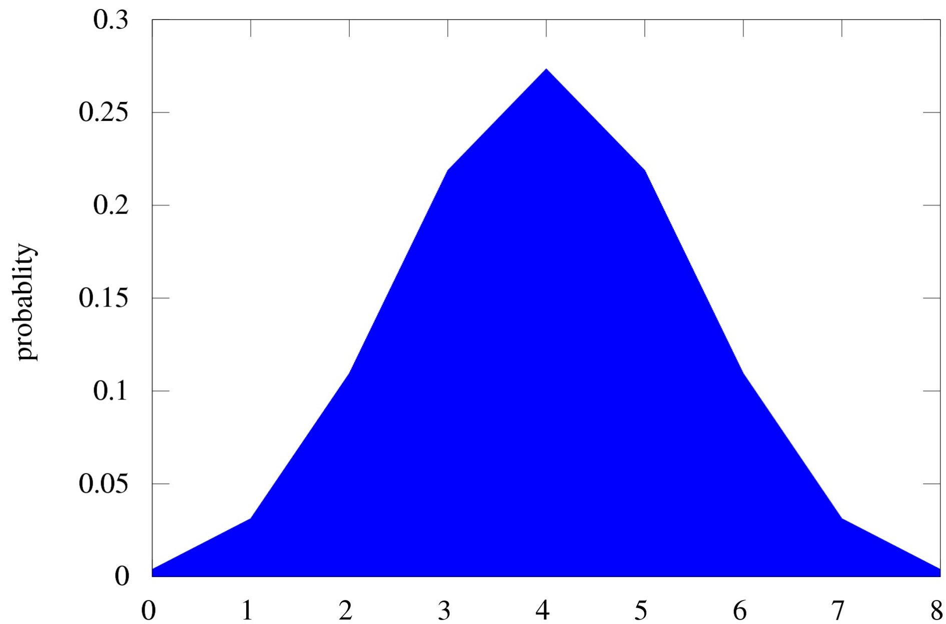
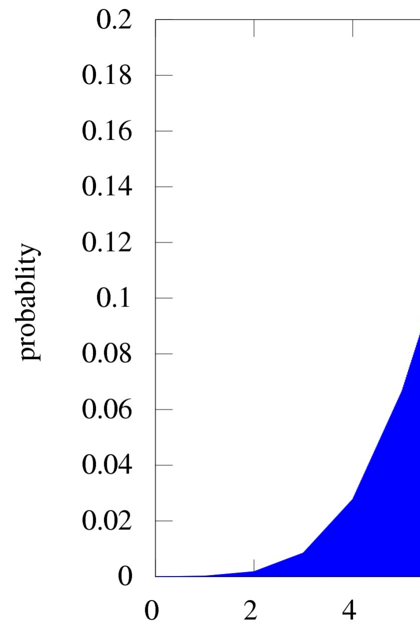
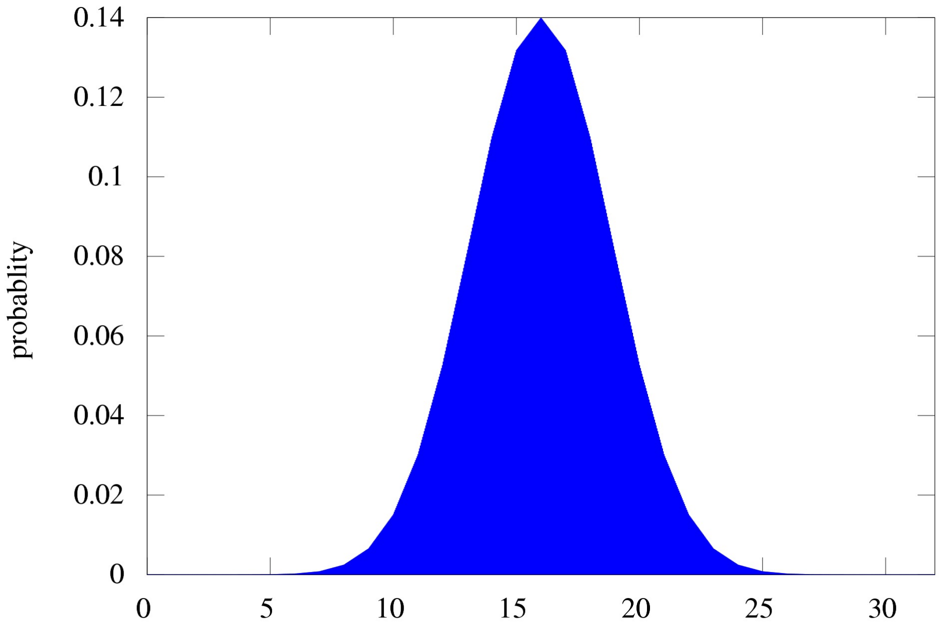
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1

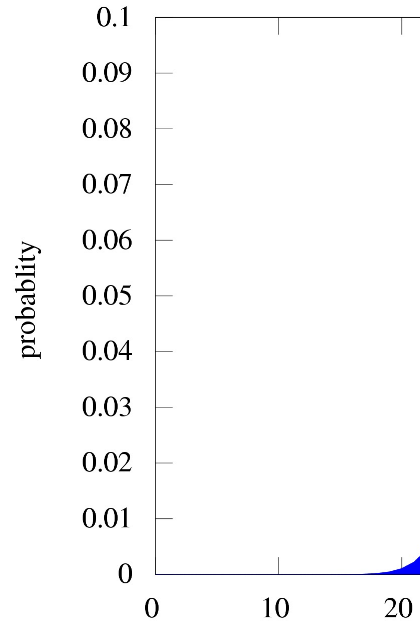

probablity
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8

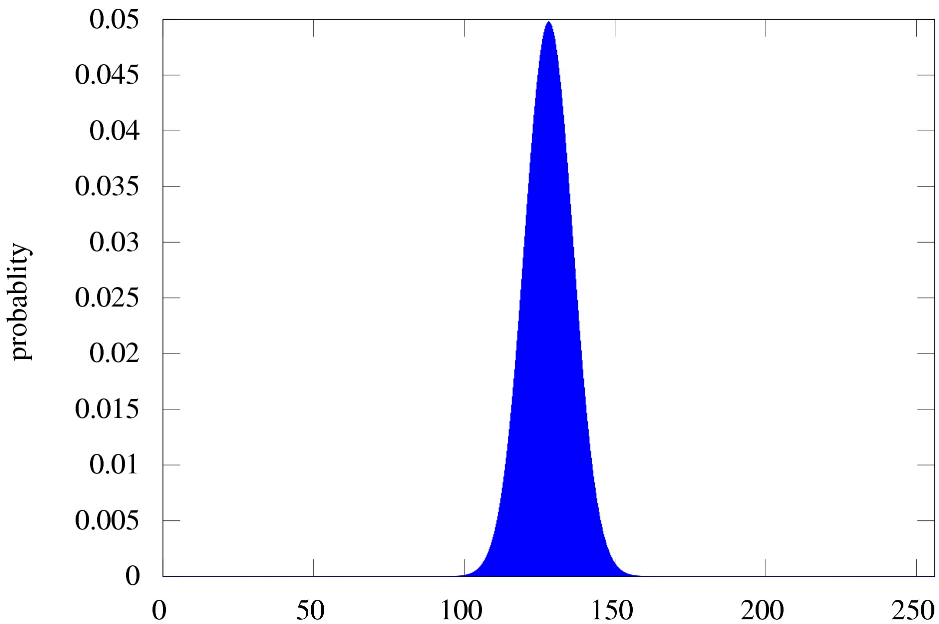
n = 8


probablity
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
2
4


probablity
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
25
30

n = 32


probablity
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
10
20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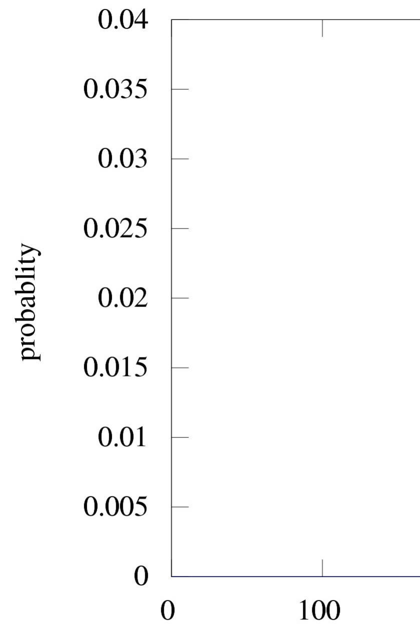
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
50
100
150
200
250

n = 256


probablity
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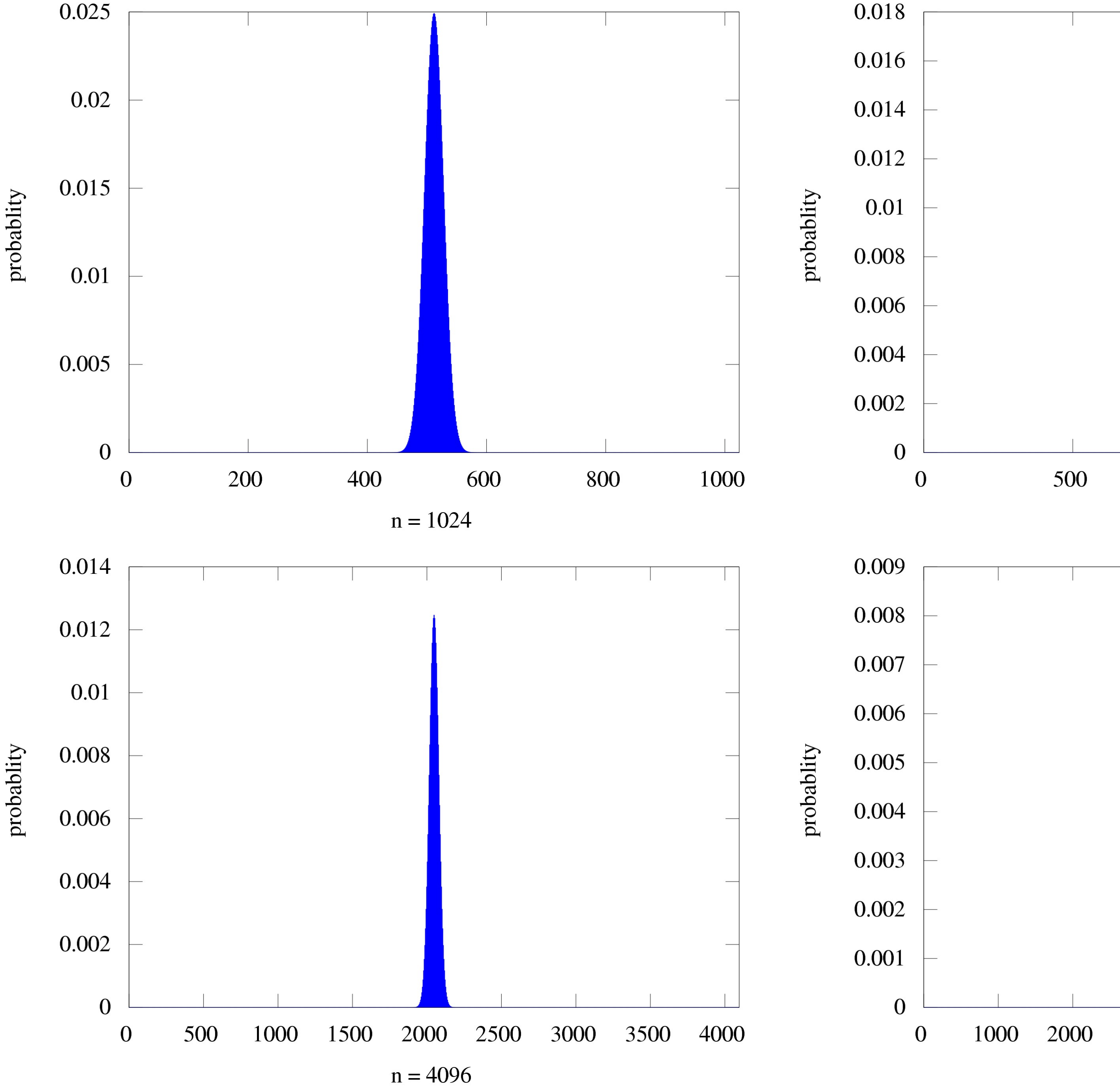
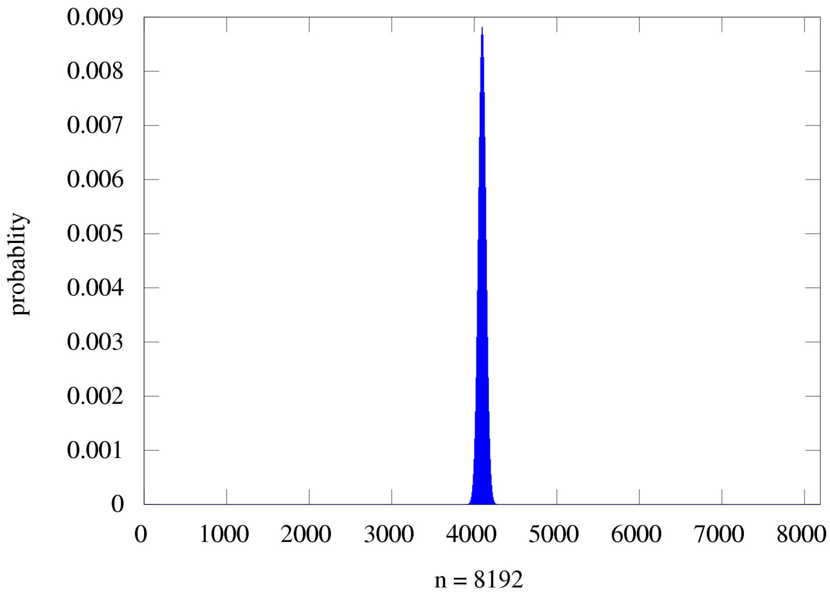
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
100


0.025
0.02
0.015
0.01
0.005
0
probablity
0
200
400
600
800
1000
n = 1024
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
probablity
0
500
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
probablity
0
500
1000
1500
2000
2500
3000
3500
4000
n = 4096
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
probablity
0
1000
2000

13.5.0.11 Massive randomness.. Is not that random.



This is known as ***concentration of mass***.

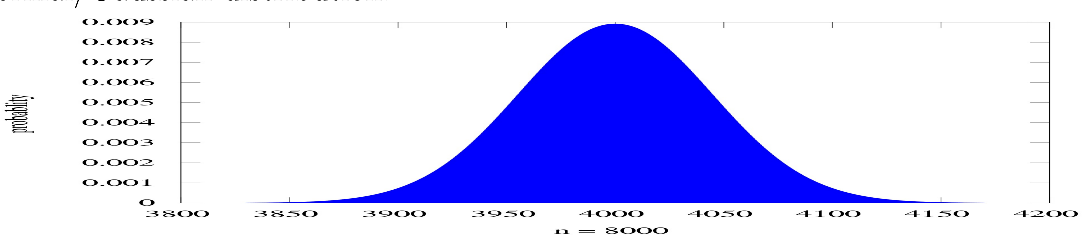
This is a very special case of the ***law of large numbers***.

13.5.1 Side note...

13.5.1.1 Law of large numbers (weakest form)...

Informal statement of law of large numbers

For n large enough, the middle portion of the binomial distribution looks like (converges to) the normal/Gaussian distribution.



13.5.1.2 Massive randomness.. Is not that random.

Intuitive conclusion

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.

13.5.1.3 Binomial distribution

X_n = numbers of heads when flipping a coin n times.

Claim

$\mathbf{Pr}\Big[X_n = i\Big] = \frac{\binom{n}{i}}{2^n}$.

Where: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Indeed, $\binom{n}{i}$ is the number of ways to choose i elements out of n elements (i.e., pick which i coin flip come up heads).

Each specific such possibility (say 0100010...) had probability $1/2^n$.

We are interested in the bad event $\mathbf{Pr}[X_n \le n/4]$ (way too few heads). We are going to prove this probability is tiny.

13.5.2 Binomial distribution

13.5.2.1 Playing around with binomial coefficients

Lemma 13.5.1. *$n! \ge (n/e)^n$.*

Proof:

$$\frac{n^n}{n!} \le \sum_{i=0}^{\infty} \frac{n^i}{i!} = e^n,$$

by the Taylor expansion of $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$. This implies that $(n/e)^n \le n!$, as required. ■

13.5.3 Binomial distribution

13.5.3.1 Playing around with binomial coefficients

Lemma 13.5.2. *For any $k \le n$, we have $\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$.*

Proof:

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} \\ &\le \frac{n^k}{k!} \le \frac{n^k}{\left(\frac{k}{e}\right)^k} = \left(\frac{ne}{k}\right)^k.\end{aligned}$$

since $k! \ge (k/e)^k$ (by previous lemma). ■

13.5.4 Binomial distribution

13.5.4.1 Playing around with binomial coefficients

$$\mathbf{Pr}\Big[X_n \le \frac{n}{4}\Big] = \sum_{k=0}^{n/4} \frac{1}{2^n}\binom{n}{k} \le \frac{1}{2^n} 2 \cdot \binom{n}{n/4}$$

For $k \leq n/4$ the above sequence behave like a geometric variable.

$$\binom{n}{k+1} \Big/ \binom{n}{k} = \frac{n!}{(k+1)!(n-k-1)!} \Big/ \frac{n!}{(k)!(n-k)!}$$
$$= \frac{n-k}{k+1} \geq \frac{(3/4)n}{n/4+1} \geq 2.$$

13.5.5 Binomial distribution

13.5.5.1 Playing around with binomial coefficients

$$\mathbf{Pr}\left[X_n \leq \frac{n}{4}\right] \leq \frac{1}{2^n} 2 \cdot \binom{n}{n/4} \leq \frac{1}{2^n} 2 \cdot \left(\frac{ne}{n/4}\right)^{n/4} \leq 2 \cdot \left(\frac{4e}{2^4}\right)^{n/4}$$
$$\leq 2 \cdot 0.68^{n/4}.$$

We just proved the following theorem.

Theorem 13.5.3. *Let X_n be the random variable which is the number of heads when flipping an unbiased coin independently n times. Then*

$$\mathbf{Pr}\left[X_n \leq \frac{n}{4}\right] \leq 2 \cdot 0.68^{n/4} \text{ and } \mathbf{Pr}\left[X_n \geq \frac{3n}{4}\right] \leq 2 \cdot 0.68^{n/4}.$$

13.6 Randomized Quick Sort and Selection

13.7 Randomized Quick Sort

13.7.0.2 Randomized QuickSort

Randomized **QuickSort**

(A) Pick a pivot element *uniformly at random* from the array.
(B) Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
(C) Recursively sort the subarrays, and concatenate them.

13.7.0.3 Example

(A) array: 16, 12, 14, 20, 5, 3, 18, 19, 1

13.7.0.4 Analysis via Recurrence

(A) Given array A of size n, let $Q(A)$ be number of comparisons of randomized **QuickSort** on A.
(B) Note that $Q(A)$ is a random variable.
(C) Let A^i_{left} and A^i_{right} be the left and right arrays obtained if:
pivot is of rank i in A.

$$Q(A) = n + \sum_{i=1}^{n} \mathbf{Pr}\Big[\text{pivot has rank } i\Big] \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}})\right).$$

Since each element of A has probability exactly of $1/n$ of being chosen:

$$Q(A) = n + \sum_{i=1}^{n} \frac{1}{n} \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}})\right).$$

13.7.0.5 Analysis via Recurrence

Let $T(n) = \max_{A:|A|=n} \mathbf{E}[Q(A)]$ be the worst-case expected running time of randomized **QuickSort** on arrays of size n.

We have, for any A:

$$Q(A) = n + \sum_{i=1}^{n} \mathbf{Pr}\Big[\text{pivot has rank } i\Big] \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}})\right)$$

Therefore, by linearity of expectation:

$$\mathbf{E}\Big[Q(A)\Big] = n + \sum_{i=1}^{n} \mathbf{Pr}\left[\begin{array}{c}\text{pivot is}\\ \text{of rank } i\end{array}\right] \left(\mathbf{E}\Big[Q(A^i_{\text{left}})\Big] + \mathbf{E}\Big[Q(A^i_{\text{right}})\Big]\right).$$

$$\Rightarrow \quad \mathbf{E}\Big[Q(A)\Big] \leq n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i)\right).$$

13.7.0.6 Analysis via Recurrence

Let $T(n) = \max_{A:|A|=n} \mathbf{E}[Q(A)]$ be the worst-case expected running time of randomized **QuickSort** on arrays of size n.

We derived:

$$\mathbf{E}\Big[Q(A)\Big] \leq n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i)\right).$$

Note that above holds for any A of size n. Therefore

$$\max_{A:|A|=n} \mathbf{E}[Q(A)] = T(n) \leq n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i)\right).$$

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$$T(n) \leq n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i) \right)$$

with base case $T(1) = 0$.

Lemma 13.7.1. $T(n) = O(n \log n)$.

Proof: (Guess and) Verify by induction. ∎

Bibliography

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