CS 473: Fundamental Algorithms, Spring 2013

Recurrences, Closest Pair and Selection

Lecture 6

February 7, 2013

Part I

Recurrences

Solving Recurrences

Two general methods:

1. Recursion tree method: need to do sums
   1. elementary methods, geometric series
   2. integration
2. Guess and Verify
   1. guessing involves intuition, experience and trial & error
   2. verification is via induction

Recurrence: Example I

1. Consider $T(n) = 2T(n/2) + n/\log n$.
2. Construct recursion tree, and observe pattern. ith level has 2^i nodes, and problem size at each node is $n/2^i$ and hence work at each node is $\frac{n}{2^i}/\log\frac{n}{2^i}$.
3. Summing over all levels

$$T(n) = \sum_{i=0}^{\log n - 1} 2^i \left[\frac{(n/2^i)}{\log(n/2^i)}\right]$$
$$= \sum_{i=0}^{\log n - 1} \frac{n}{\log n - i}$$
$$= n \sum_{j=1}^{\log n} \frac{1}{j} = nH_{\log n} = \Theta(n \log\log n)$$

Recurrence: Example II

1. Consider...
2. What is the depth of recursion? $\sqrt{n}, \sqrt{\sqrt{n}}, \sqrt{\sqrt{\sqrt{n}}}, \ldots, O(1)$.
3. Number of levels: $n^{2^{-L}} = 2$ means $L = \log \log n$.
4. Number of children at each level is 1, work at each node is 1
5. Thus, $T(n) = \sum_{i=0}^{L} 1 = \Theta(L) = \Theta(\log \log n)$.

Recurrence: Example III

1. Consider $T(n) = \sqrt{n}T(\sqrt{n}) + n$.
2. Using recursion trees: number of levels $L = \log \log n$
3. Work at each level? Root is n, next level is $\sqrt{n} \times \sqrt{n} = n$, so on. Can check that each level is n.
4. Thus, $T(n) = \Theta(n \log \log n)$

Recurrence: Example IV

1. Consider $T(n) = T(n/4) + T(3n/4) + n$.
2. Using recursion tree, we observe the tree has leaves at different levels (a *lop-sided* tree).
3. Total work in any level is at most n. Total work in any level without leaves is exactly n.
4. Highest leaf is at level $\log_4 n$ and lowest leaf is at level $\log_{4/3} n$
5. Thus, $n \log_4 n \leq T(n) \leq n \log_{4/3} n$, which means $T(n) = \Theta(n \log n)$

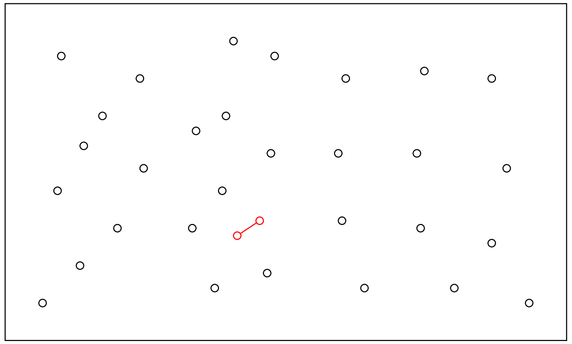
Part II

Closest Pair

Closest Pair - the problem

Input Given a set $\mathbf{S}$ of $\mathbf{n}$ points on the plane

Goal Find $\mathbf{p}, \mathbf{q} \in \mathbf{S}$ such that $\mathbf{d}(\mathbf{p}, \mathbf{q})$ is minimum



Applications

1. Basic primitive used in graphics, vision, molecular modelling
2. Ideas used in solving nearest neighbor, Voronoi diagrams, Euclidean MST

Algorithm: Brute Force

1. Compute distance between every pair of points and find minimum.
2. Takes $\mathbf{O}(\mathbf{n}^2)$ time.
3. Can we do better?

Closest Pair: 1-d case

Input Given a set $\mathbf{S}$ of $\mathbf{n}$ points on a line

Goal Find $\mathbf{p}, \mathbf{q} \in \mathbf{S}$ such that $\mathbf{d}(\mathbf{p}, \mathbf{q})$ is minimum

Algorithm

1. Sort points based on coordinate
2. Compute the distance between successive points, keeping track of the closest pair.

Running time $\mathbf{O}(\mathbf{n} \log \mathbf{n})$

Can we do this in better running time?
Can reduce Distinct Elements Problem (see lecture 1) to this problem in $\mathbf{O}(\mathbf{n})$ time. Do you see how?

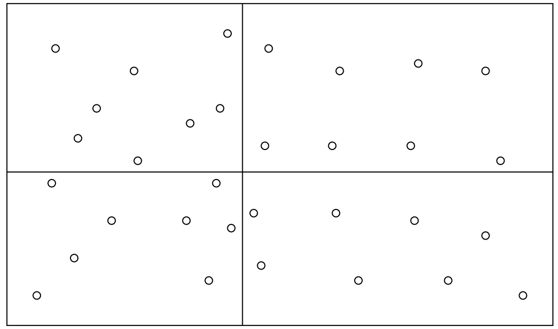
Generalizing 1-d case

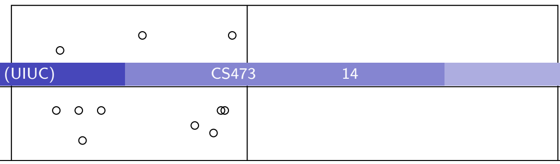
Can we generalize **1**-d algorithm to **2**-d?
Sort according to **x** or **y**-coordinate??
No easy generalization.

First Attempt

Divide and Conquer I

1. Partition into 4 quadrants of roughly equal size.Not always!
2. Find closest pair in each quadrant recursively
3. Combine solutions

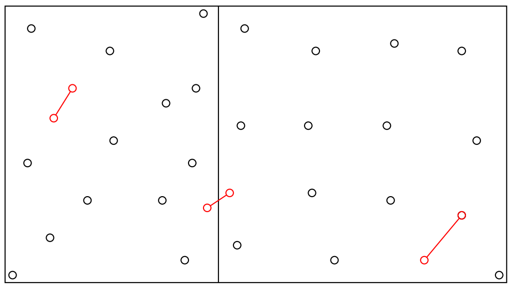




New Algorithm

Divide and Conquer II

1. Divide the set of points into two equal parts via vertical line
2. Find closest pair in each half recursively
3. Find closest pair with one point in each half
4. Return the best pair among the above 3 solutions



New Algorithm

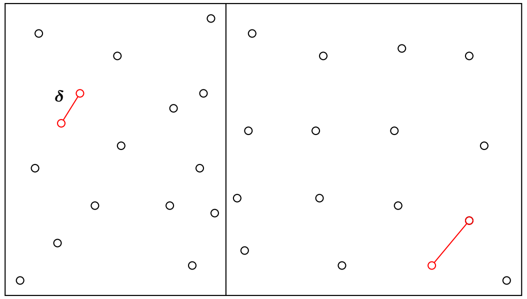
Divide and Conquer II

1. Divide the set of points into two equal parts via vertical line
2. Find closest pair in each half recursively
3. Find closest pair with one point in each half
4. Return the best pair among the above 3 solutions

1. Sort points based on **x**-coordinate and pick the median. Time $= O(n \log n)$
2. How to find closest pair with points in different halves? $O(n^2)$ is trivial. Better?

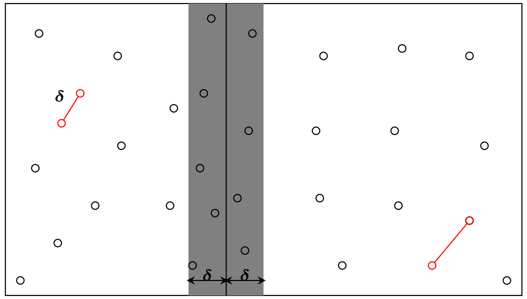
Combining Partial Solutions

1. Does it take $\mathbf{O(n^2)}$ to combine solutions?
2. Let δ be the distance between closest pairs, where both points belong to the same half.

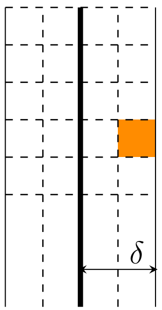


Combining Partial Solutions

1. Let δ be the distance between closest pairs, where both points belong to the same half.
2. Need to consider points within δ of dividing line



Sparsity of Band XXX



Divide the band into square boxes of size $\delta/2$

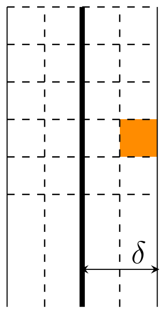
Lemma

Each box has at most one point

Proof.

If not, then there are a pair of points (both belonging to one half) that are at most $\sqrt{2}\delta/2 < \delta$ apart! □

Searching within the Band



Lemma

Suppose $\mathbf{a, b}$ *are both in the band* $\mathbf{d(a, b)} < \delta$ *then* $\mathbf{a, b}$ *have at most two rows of boxes between them.*

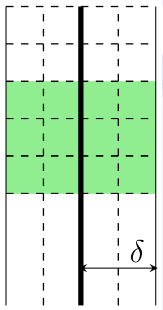
Proof.

Each row of boxes has height $\delta/2$. If more than two rows then $\mathbf{d(a, b)} > 2 \cdot \delta/2$! □

Searching within the Band

Corollary

Order points according to their $\mathbf{y}$*-coordinate. If* $\mathbf{p}, \mathbf{q}$ *are such that* $\mathbf{d(p, q)} < \delta$ *then* $\mathbf{p}$ *and* $\mathbf{q}$ *are within* $\mathbf{11}$ *positions in the sorted list.*



Proof.

1. $\leq \mathbf{2}$ points between them if $\mathbf{p}$ and $\mathbf{q}$ in same row.
2. $\leq \mathbf{6}$ points between them if $\mathbf{p}$ and $\mathbf{q}$ in two consecutive rows.
3. $\leq \mathbf{10}$ points between them if $\mathbf{p}$ and $\mathbf{q}$ one row apart.
4. $\implies$ More than ten points between them in the sorted $\mathbf{y}$ order than $\mathbf{p}$ and $\mathbf{q}$ are more than two rows apart.
5. $\implies \mathbf{d(p, q)} > \delta$. A contradiction. ∎

The Algorithm

```
ClosestPair(P):
   1. Find vertical line L splits P into equal halves: P1 and P2
   2. δ1 ← ClosestPair(P1).
   3. δ2 ← ClosestPair(P2).
   4. δ = min(δ1, δ2)
   5. Delete points from P further than δ from L
   6. Sort P based on y-coordinate into an array A
   7. for i = 1 to |A| − 1 do
         for j = i + 1 to min{i + 11, |A|} do
            if (dist(A[i], A[j]) < δ) update δ and closest pair
```

1. Step 1, involves sorting and scanning. Takes $\mathbf{O(n \log n)}$ time.
2. Step 5 takes $\mathbf{O(n)}$ time.
3. Step 6 takes $\mathbf{O(n \log n)}$ time
4. Step 7 takes $\mathbf{O(n)}$ time.

Running Time

The running time of the algorithm is given by

$$\mathbf{T(n) \leq 2T(n/2) + O(n \log n)}$$

Thus, $\mathbf{T(n) = O(n \log^2 n)}$.

Improved Algorithm

Avoid repeated sorting of points in band: two options

1. Sort all points by $\mathbf{y}$-coordinate and store the list. In conquer step use this to avoid sorting
2. Each recursive call returns a list of points sorted by their $\mathbf{y}$-coordinates. Merge in conquer step in linear time.

Analysis: $\mathbf{T(n) \leq 2T(n/2) + O(n) = O(n \log n)}$

Part III

Selecting in Unsorted Lists

Quick Sort

Quick Sort [Hoare]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself. Linear scan of array does it. Time is $O(n)$
3. Recursively sort the subarrays, and concatenate them.

Example:

1. array: 16, 12, 14, 20, 5, 3, 18, 19, 1
2. pivot: 16
3. split into 12, 14, 5, 3, 1 and 20, 19, 18 and recursively sort
4. put them together with pivot in middle

Time Analysis

1. Let k be the rank of the chosen pivot. Then, $T(n) = T(k-1) + T(n-k) + O(n)$
2. If $k = \lceil n/2 \rceil$ then $T(n) = T(\lceil n/2 \rceil - 1) + T(\lfloor n/2 \rfloor) + O(n) \leq 2T(n/2) + O(n)$. Then, $T(n) = O(n \log n)$.
   1. Theoretically, median can be found in linear time.
3. Typically, pivot is the first or last element of array. Then,

$$T(n) = \max_{1 \leq k \leq n} (T(k-1) + T(n-k) + O(n))$$

In the worst case $T(n) = T(n-1) + O(n)$, which means $T(n) = O(n^2)$. Happens if array is already sorted and pivot is always first element.

Problem - Selection

Input: Unsorted array A of n integers

Goal: Find the jth smallest number in A (*rank* j number)

Example

$A = \{4, 6, 2, 1, 5, 8, 7\}$ and $j = 4$. The jth smallest element is 5.

Median: $j = \lfloor (n+1)/2 \rfloor$

Algorithm I

1. Sort the elements in A
2. Pick jth element in sorted order

Time taken $= O(n \log n)$

Do we need to sort? Is there an $O(n)$ time algorithm?

Algorithm II

If $\mathbf{j}$ is small or $\mathbf{n} - \mathbf{j}$ is small then

1. Find $\mathbf{j}$ smallest/largest elements in $\mathbf{A}$ in $\mathbf{O(jn)}$ time. (How?)
2. Time to find median is $\mathbf{O(n^2)}$.

Divide and Conquer Approach

1. Pick a pivot element $\mathbf{a}$ from $\mathbf{A}$
2. Partition $\mathbf{A}$ based on $\mathbf{a}$.
   $\mathbf{A_{less}} = \{\mathbf{x} \in \mathbf{A} \mid \mathbf{x} \leq \mathbf{a}\}$ and $\mathbf{A_{greater}} = \{\mathbf{x} \in \mathbf{A} \mid \mathbf{x} > \mathbf{a}\}$
3. $|\mathbf{A_{less}}| = \mathbf{j}$: return $\mathbf{a}$
4. $|\mathbf{A_{less}}| > \mathbf{j}$: recursively find $\mathbf{j}$th smallest element in $\mathbf{A_{less}}$
5. $|\mathbf{A_{less}}| < \mathbf{j}$: recursively find $\mathbf{k}$th smallest element in $\mathbf{A_{greater}}$ where $\mathbf{k} = \mathbf{j} - |\mathbf{A_{less}}|$.

Time Analysis

1. Partitioning step: $\mathbf{O(n)}$ time to scan $\mathbf{A}$
2. How do we choose pivot? Recursive running time?

Suppose we always choose pivot to be $\mathbf{A[1]}$.

Say $\mathbf{A}$ is sorted in increasing order and $\mathbf{j} = \mathbf{n}$.
Exercise: show that algorithm takes $\mathbf{\Omega(n^2)}$ time

A Better Pivot

Suppose pivot is the ℓth smallest element where $\mathbf{n/4} \leq \ell \leq \mathbf{3n/4}$.
That is pivot is *approximately* in the middle of $\mathbf{A}$
Then $\mathbf{n/4} \leq |\mathbf{A_{less}}| \leq \mathbf{3n/4}$ *and* $\mathbf{n/4} \leq |\mathbf{A_{greater}}| \leq \mathbf{3n/4}$. If we apply recursion,

$$\mathbf{T(n) \leq T(3n/4) + O(n)}$$

Implies $\mathbf{T(n) = O(n)}$!

How do we find such a pivot? Randomly? In fact works!
Analysis a little bit later.

Can we choose pivot deterministically?

Divide and Conquer Approach

A game of medians

Idea

1. Break input $\mathbf{A}$ into many subarrays: $\mathbf{L_1, \ldots L_k}$.
2. Find median $\mathbf{m_i}$ in each subarray $\mathbf{L_i}$.
3. Find the median $\mathbf{x}$ of the medians $\mathbf{m_1, \ldots, m_k}$.
4. Intuition: The median $\mathbf{x}$ should be close to being a good median of all the numbers in $\mathbf{A}$.
5. Use $\mathbf{x}$ as pivot in previous algorithm.

But we have to be...

More specific...

1. Size of each group?
2. How to find median of medians?

Choosing the pivot

A clash of medians

1. Partition array $\mathbf{A}$ into $\lceil \mathbf{n/5} \rceil$ lists of $\mathbf{5}$ items each.
   $\mathbf{L_1 = \{A[1], A[2], \ldots, A[5]\}}$, $\mathbf{L_2 = \{A[6], \ldots, A[10]\}}$, $\ldots$,
   $\mathbf{L_i = \{A[5i+1], \ldots, A[5i-4]\}}$, $\ldots$,
   $\mathbf{L_{\lceil n/5 \rceil} = \{A[5\lceil n/5 \rceil - 4, \ldots, A[n]\}}$.
2. For each $\mathbf{i}$ find median $\mathbf{b_i}$ of $\mathbf{L_i}$ using brute-force in $\mathbf{O(1)}$ time. Total $\mathbf{O(n)}$ time
3. Let $\mathbf{B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}}$
4. Find median $\mathbf{b}$ of $\mathbf{B}$

Lemma

Median of $\mathbf{B}$ *is an* approximate *median of* $\mathbf{A}$*. That is, if* $\mathbf{b}$ *is used a pivot to partition* $\mathbf{A}$*, then* $|\mathbf{A}_{less}| \leq \mathbf{7n/10 + 6}$ and $|\mathbf{A}_{greater}| \leq \mathbf{7n/10 + 6}$.

Algorithm for Selection

A storm of medians

```
select(A, j):
    Form lists L_1, L_2, ..., L_⌈n/5⌉ where L_i = {A[5i − 4], ..., A[5i]}
    Find median b_i of each L_i using brute-force
    Find median b of B = {b_1, b_2, ..., b_⌈n/5⌉}
    Partition A into A_less and A_greater using b as pivot
    if (|A_less|) = j return b
    else if (|A_less|) > j)
        return select(A_less, j)
    else
        return select(A_greater, j − |A_less|)
```

How do we find median of $\mathbf{B}$? Recursively!

Running time of deterministic median selection

A dance with recurrences

$$\mathbf{T(n) = T(\lceil n/5 \rceil) + \max\{T(|A_{less}|), T(|A_{greater})|\} + O(n)}$$

From Lemma,

$$\mathbf{T(n) \leq T(\lceil n/5 \rceil) + T(\lfloor 7n/10 + 6 \rfloor) + O(n)}$$

and

$$\mathbf{T(1) = 1}$$

Exercise: show that $\mathbf{T(n) = O(n)}$

Median of Medians: Proof of Lemma

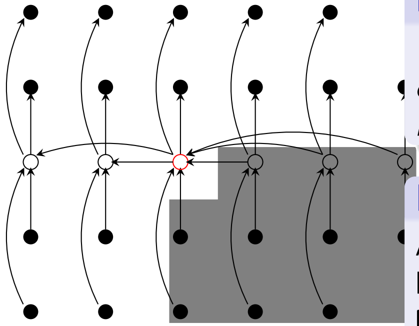


Figure: Shaded elements are all greater than $\mathbf{b}$

Proposition

There are at least $\mathbf{3n/10 - 6}$ *elements greater than the median of medians* $\mathbf{b}$.

Proof.

At least half of the $\lceil \mathbf{n/5} \rceil$ groups have at least 3 elements larger than $\mathbf{b}$, except for last group and the group containing $\mathbf{b}$. So $\mathbf{b}$ is less than

$$3(\lceil (1/2) \lceil n/5 \rceil \rceil - 2) \geq 3n/10 - 6$$

□

Median of Medians: Proof of Lemma

Proposition

There are at least $\mathbf{3n/10 - 6}$ *elements greater than the median of medians* $\mathbf{b}$.

Corollary

$|\mathbf{A}_{less}| \leq \mathbf{7n/10 + 6}$.

Via symmetric argument,

Corollary

$|\mathbf{A}_{greater}| \leq \mathbf{7n/10 + 6}$.

Questions to ponder

1. Why did we choose lists of size $\mathbf{5}$? Will lists of size $\mathbf{3}$ work?
2. Write a recurrence to analyze the algorithm's running time if we choose a list of size $\mathbf{k}$.

Median of Medians Algorithm

Due to:
M. Blum, R. Floyd, D. Knuth, V. Pratt, R. Rivest, and R. Tarjan.
"Time bounds for selection".
Journal of Computer System Sciences (JCSS), 1973.

How many Turing Award winners in the author list?
All except Vaughn Pratt!

Takeaway Points

1. Recursion tree method and guess and verify are the most reliable methods to analyze recursions in algorithms.
2. Recursive algorithms naturally lead to recurrences.
3. Some times one can look for certain type of recursive algorithms (reverse engineering) by understanding recurrences and their behavior.