HW 10 (due Monday, at noon, April 22, 2013)

CS 473: Fundamental Algorithms, Spring 2013 Version: **1.02**

Make sure that you write the solutions for the problems on separate sheets of paper. Write your name and netid on each sheet.

Collaboration Policy: The homework can be worked in groups of up to 3 students each.

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1. (45 PTS.) Coolest path

After Kris moved to Illinois, he started dating a computer scientist vampire climber girl Oinoe that he met at the climbing gym. Kris is a multi-dimensional personality and apart from climbing he also enjoys and excels at snowboarding. To his pleasant surprise, Oinoe was also a decent snowboarder and so they decided to take a trip to the backcountry mountains of New Zealand. They packed a few days worth of supplies, got their snowboarding gear on and hired a helicopter from the mountain base (hereafter referred to as the Base) to drop them off at the peak of a far away mountain called Death Peak. Kris and Oinoe's goal was to make their way from Death Peak back to the Base. The mountain was covered in powder and there was nobody else on it. The only thing that would reassure them that they were on a path back to the Base were some red poles planted in the snow at various parts of the mountain, called Stations. We can view the mountain as an undirected graph $G = (V, E)$ where each node is a Station and an edge (u, v) indicated that one can travel directly from station u to station v by snowboard (Kris and Oinoe carried with them a new kind of snowboard which was enhanced by a motor and allowed them to trave through flat parts of the mountain easily). The Death Peak is represented by a node s and the Base by a node t. Each edge e has a length $l_e \geq 0$ which represents distance from one Station to another. Also, some edges represent paths that are higher risk than others, in the sense that they are more avalanche-prone, have more tree-wells or obstacles along the way. So each edge e also has an integer risk $r_e \geq 0$, indicating the expected amount of damage in their health or equipment, if one traverses this edge.

It would be safest to travel by traversing the ridge of the mountain till they reach the end of the Sierra and then go downhill a very easy slope, but that would take them many days and they will run out of food. It would be fastest to just go down the steepest slope from Death Peak to the base of the mountain but that is very dangerous to create an avalanche. In general, for every path p from s to t, we define its total length to be the sum of the lengths of all its edges and its total risk to be the sum of the risks of all its edges.

Kris and Oinoe are looking for a complex type of shortest path in that graph that they name the *Coolest Path*: they need to get from s to t along a path whose total length and total risk is reasonably small. In concrete terms, the problem they want to solve is the following: given a graph with lengths and risks as above and integers L and R, is there a path from s to t whose total length is at most L and whose total risk is at most R?

Show that the Coolest Path problem is NP-Complete.

2. (30 PTS.) Brooklyn is learning how to speak

Your friend's pre-school age daughter Brooklyn has recently learned to spell some simple words. To help encourage this, her parents got her a colorful set of refrigerator magnets featuring the letters of the alphabet (some number of copies of each letter), and the last time you saw her, the two of you spent a while arranging the magnets to spell out words that she knows.

Somehow with you and Brooklyn, things end up getting more elaborate than originally planned, and soon the two of you were trying to spell out words so as to use up all the magnets int he full set-that is, picking words that she knows how to spell, so that they were all spelled out, each magnet was participating in the spelling of exactly one word. Multiple copies of words are okay here.

This turned out to be pretty difficult, and it was only later that you realized a plausible reason for this. Suppose we consider a general version of the problem of *Using Up All the Refrigerator Magnets*, where we replace the English alphabet by an arbitrary collection of symbols, and we model Brooklyn's vocabulary as an arbitrary set of strings over this collection of symbols. The goal is the same. Prove that the problem of *Using Up All the Refrigerator Magnets* is NP-Complete.

3. (25 PTS.) Path Selection

Consider the following problem. You are managing a communication network, modeled by a directed graph $G = (V, E)$. There are c *users* who are interested in making use of this network. User i (for each $i = 1, 2, \ldots, c$) issues a *request* to reserve a specific path P_i in G on which to transmit data.

You are interested in accepting as many of these path requests as possible, subject to the following restriction: if you accept both P_i and P_j, then P_i and P_j can not share any nodes.

Thus the *Path Selection Problem* asks: Given a directed graph $G = (V, E)$, a set of requests $P_1, \ldots, P_c$-each of which must be a path in G- and a number k, is it possible to select at least k of the paths so that no two of the selected paths share any nodes?

Find a polynomial time algorithm for Path Selection or show that the problem is NP-Complete.