HW 8 (due Monday, at noon, April 8, 2013)

CS 473: Fundamental Algorithms, Spring 2013 Version: **1.01**

Make sure that you write the solutions for the problems on separate sheets of paper. Write your name and netid on each sheet.

Collaboration Policy: The homework can be worked in groups of up to 3 students each.

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1. (30 PTS.) Kris and the Climbing Vampires of CS @ Illinois

After Kris (your well-known professional climber) got tired of living in Colorado, he moved to Champaign, Illinois. Once the climbing community of this little town found out that Kris was living here, they felt honored and elected him president of "CS @ Illinois Climbing Club", also known as "The Club". The Computer Science Department at UIUC has n (semi-professional, vampire) climbers, who are members of The Club. In order for them to retain their membership, they need to participate in various competitions yearly. There are m competitions each year and the j'th competition needs k_j participants from The Club. Kris, trying to fulfil his president duties, asked each member to volunteer to participate in a few competitions. Let $S_i \subseteq \{1, 2, \ldots, m\}$ be the set of competitions that a computer scientist vampire climber i has volunteered for. A competition assignment consists of sets $S'_1, S'_2, \ldots, S'_n$ where $S'_i \subseteq \{1, 2, \ldots, m\}$ is the set of competitions that club member i will participate in. A *valid* competition assignment has to satisfy two constraints: (i) for each club member i, $S'_i \subseteq S_i$, that is each member is only participating in competitions that he/she has volunteered for, and (ii) each competition j has k_j club members assigned to it, or in other words j occurs in at least k_j of the sets $S'_1, S'_2, \ldots, S'_n$. Kris noticed that often there is no valid competition assignment because computer scientist vampire climbers naturally are inclined to volunteer for as few competitions as possible (stemming from the lazy nature of computer scientists and vampires). To overcome this, the definition of a valid assignment is relaxed as follows. Let ℓ be some integer. An assignment $S'_1, S'_2, \ldots, S'_n$ is now said to be valid if (i) $|S'_i - S_i| \leq \ell$ and (ii) each competition j has k_j club members participating at it. The new condition (i) means that a member i may participate up to ℓ competitions not on the list S_i that he/she volunteered for. Describe an algorithm to check if there is a valid competition assignment with the relaxed definition.

2. (40 PTS.) Augmenting Paths in Residual Networks.

You are given an integral instance G of network flow. Let C be the value of the maximum flow in G.

(A) (8 PTS.) Given a flow f in G, and its residual network G_f, describe how to compute, as fast as possible, the highest capacity augmenting path flow from s to t. Prove the correctness of your algorithm.

(B) (8 PTS.) Prove, that if the maximum flow in G_f has value T, then the augmenting path you found in (A) has capacity at least T/m.

(C) (8 PTS.) Consider the algorithm that starts with the empty flow f, and repeatedly applies (A) to G_f (recomputing it after each iteration) until s and t are disconnected. Prove that this algorithm computes the maximum flow in G.

(D) (8 PTS.) Consider the algorithm from (C), and the flow g it computes after m iterations. Prove that $|g| >= C/10$ (here 10 is not tight).

(E) (8 PTS.)Give a bound, as tight as possible, on the running time of your algorithm, as a function of n, m, and C.

3. (30 PTS.) Edge-Disjoint Paths.

You are given a directed graph $G = (V, E)$ and a natural number k.

(A) We can define a relation $\to_{G,k}$ on pairs of vertices of G as follows. If $x, y \in V$ we say that $x \to_{G,k} y$ if there exist k mutually edge disjoint paths from x to y in G. Is it true that for every G and every $k \geq 0$ the relation $\to_{G,k}$ is transitive? That is, is it always the case that if $x \to_{G,k} y$ and $y \to_{G,k} z$ then we have $x \to_{G,k} z$? Give a proof or a counterexample.

(B) Suppose that for each node v of G the number of edges into v is equal to the number of edges out of v. Let x, y be two nodes and suppose there exist k mutually edge-disjoint paths from x to y. Does it follow that there exist k mutually disjoint paths from y to x? Give a proof or a counterexample.