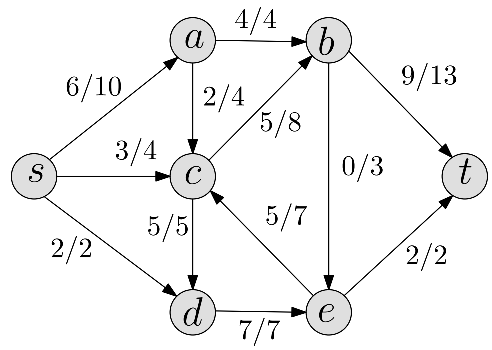
CS 473: Fundamental Algorithms, Spring 2013

Discussion 10

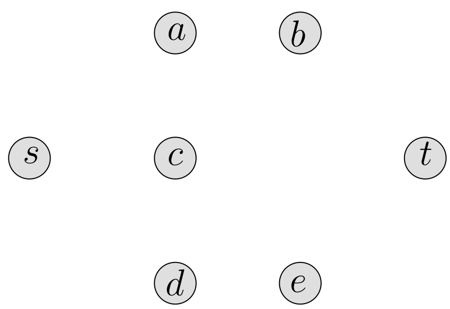
March 27, 2013

10.1. Go With the Flow.

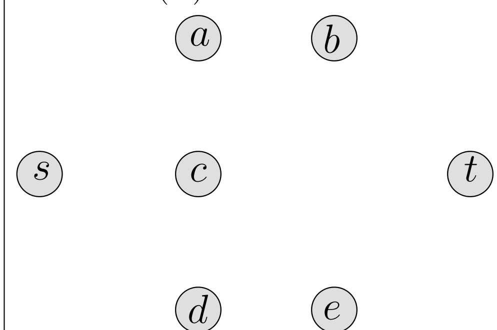
The figure on the right shows a flow network along with a flow. In the figure, the notation α/β for an edge means that the flow on the edge is α and the capacity of the edge is β.



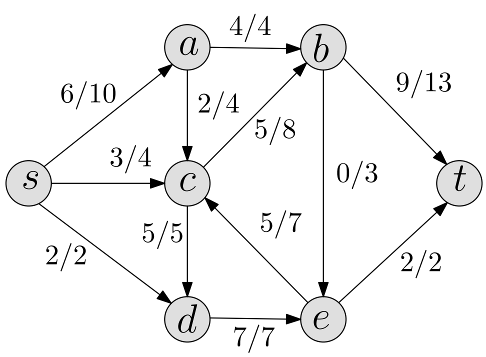
(A) What is the value of the given flow? Is it a maximum flow? Show the residual graph for the above graph and flow in the figure below.

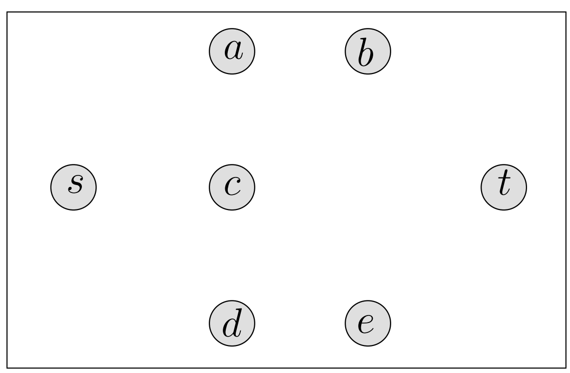


(B) Show an $s-t$ path in the residual graph and state its bottleneck capacity. You only need to draw the path from the graph you showed in (A).



(C) Show the new flow on the *original* graph after augmenting on the path you found in (B). Use the notation a/b to indicate the flow on an edge and its capacity.





(D) What is the capacity of a minimum-cut in the given graph? Find a cut with that capacity.

10.2. Capacities on Nodes.

In a standard $s-t$ maximum flow problem, we assume that edges have capacities, and there is no limit on how much flow is allowed to pass through a node. In this problem, we consider the variant where nodes have capacities.

Let $G = (V, E)$ be a directed graph with source s and sink t. Let $c : V \to \Re^{+}$ be a capacity function. Recall that a flow f assigns a flow value $f(e)$ to each *edge* e. A flow f is *feasible* if the total flow into every vertex v is at most $c(v)$:

$$f^{\text{in}}(v) \leq c(v) \qquad \text{for every vertex } v.$$

Describe a reduction from the problem of finding a feasible $s - t$ flow of maximum value in G to the standard flow problem with edge capacities.

10.3. TRANSITIVITY OF MIN-CUT VALUES.
Let $\mathsf{G} = (V, E)$ be a directed graph with non-negative edge-capacities $c : E \to R^{+}$. For any two distinct nodes $x, y \in V$ let $\alpha_{\mathsf{G}}(x, y)$ be the capacity of the minimum-cut between a and b in G. For any three distinct nodes $u, v, w \in V$ prove the following:

$$\alpha_{\mathsf{G}}(u, v) \geq \min\left\{\alpha_{\mathsf{G}}(u, w),\, \alpha_{\mathsf{G}}(w, v)\right\}.$$

10.4. CAPACITATION, YEH, YEH, YEH.
Suppose you are given a directed graph $G = (V, E)$, with a positive integer capacity c_e on each edge e, a designated source $s \in V$, and a designated sink $t \in V$. You are also given a maximum s-t flow in G, defined by a flow value f_e on each edge e. You can assume the flow is integral. The flow $\{f_e\}$ is ***acyclic***: There is no cycle in G on which all edges carry positive flow.
Now suppose we pick a specific edge $e^{*} \in E$ and reduce its capacity by 1 unit. Show how to find a maximum flow in the resulting capacitated graph in time $O(m + n)$, where m is the number of edges in G and n is the number of nodes.

10.5. RESIDUAL GRAPH PROPERTIES.
Prove the following property about residual graphs:

Claim 10.1. *Let f be a flow in G and G_f be the residual graph. If f' is a flow in G_f, then $f'' = f + f'$ is a flow in G of value $v(f) + v(f')$.*

Here, for each edge $e = (u, v)$ in G_f, if e is a forward edge, then $f''(e) = f(e) + f'(e)$. If e is a backward edge, then let $e' = (v, u)$ and define $f''(e') = f(e') - f'(e)$.