1. We say that an array $A[1..n]$ is *k-sorted* if it can be divided into k blocks, each of size n/k, such that the elements in each block are larger than the elements in earlier blocks, and smaller than elements in later blocks. The elements within each block need not be sorted.

   For example, the following array is 4-sorted:

1	2	4	3	7	6	8	5	10	11	9	12	15	13	16	14

   (a) Describe an algorithm that k-sorts an arbitrary array in time $O(n \log k)$.

   (b) Prove that any comparison-based k-sorting algorithm requires $\Omega(n \log k)$ comparisons in the worst case.

   (c) Describe an algorithm that completely sorts an already k-sorted array in time $O(n \log(n/k))$.

   (d) Prove that any comparison-based algorithm to completely sort a k-sorted array requires $\Omega(n \log(n/k))$ comparisons in the worst case.

   In all cases, you can assume that n/k is an integer and that $n! \approx \left(\frac{n}{e}\right)^n$.

2. Recall the nuts and bolts problem from the first randomized algorithms lecture. You are given n nuts and n bolts of different sizes. Each nut matches exactly one bolt and vice versa. The nuts and bolts are all almost exactly the same size, so we can't tell if one bolt is bigger than the other, or if one nut is bigger than the other. If we try to match a nut with a bolt, however, we will discover either that the nut is too big, the nut is too small, or the nut is just right for the bolt. The goal was to find the matching nut for every bolt.

   Now consider a relaxed version of the problem where the goal is to find the matching nuts for *half* of the bolts, or equivalently, to find $n/2$ matched nut-bolt pairs. (It doesn't matter *which* $n/2$ nuts and bolts are matched.) Prove that any deterministic algorithm to solve this problem must perform $\Omega(n \log n)$ nut-bolt tests in the worst case.

3. UIUC has just finished constructing the new Reingold Building, the tallest dormitory on campus. In order to determine how much insurance to buy, the university administration needs to determine the highest safe floor in the building. A floor is consdered *safe* if ~~a drunk student~~ ***an egg*** can fall from a window on that floor and land without breaking; if the egg breaks, the floor is considered *unsafe*. Any floor that is higher than an unsafe floor is also considered unsafe. The only way to determine whether a floor is safe is to drop an egg from a window on that floor.

   You would like to find the lowest unsafe floor L by performing as few tests as possible; unfortunately, you have only a very limited supply of eggs.

   (a) Prove that if you have only one egg, you can find the lowest unsafe floor with L tests. *[Hint: Yes, this is trivial.]*

   (b) Prove that if you have only one egg, you must perform at least L tests in the worst case. In other words, prove that your algorithm from part (a) is optimal. *[Hint: Use an adversary argument.]*

   (c) Describe an algorithm to find the lowest unsafe floor using *two* eggs and only $O(\sqrt{L})$ tests. *[Hint: Ideally, each egg should be dropped the same number of times. How many floors can you test with n drops?]*

   (d) Prove that if you start with two eggs, you must perform at least $\Omega(\sqrt{L})$ tests in the worst case. In other words, prove that your algorithm from part (c) is optimal.