This exam lasts 120 minutes.
Write your answers in the separate answer booklet.
Please return this question sheet with your answers.

1. Each of these ten questions has one of the following five answers:

   A: $\Theta(1)$ B: $\Theta(\log n)$ C: $\Theta(n)$ D: $\Theta(n \log n)$ E: $\Theta(n^2)$

   Choose the correct answer for each question. Each correct answer is worth $+1$ point; each incorrect answer is worth $-1/2$ point; and each "I don't know" is worth $+1/4$ point. Negative scores will be recorded as 0.

   (a) What is $\dfrac{n}{4} + \dfrac{4}{n}$?

   (b) What is $\displaystyle\sum_{i=1}^{n} \frac{i}{n^2}$?

   (c) What is $\sqrt{\displaystyle\sum_{i=1}^{n} i}$?

   (d) How many digits are required to write the number $n!$ (the factorial of n) in decimal?

   (e) What is the solution to the recurrence $E(n) = E(n-3) + \pi$?

   (f) What is the solution to the recurrence $F(n) = 4F(n/2) + 6n$?

   (g) What is the solution to the recurrence $G(n) = 9G(n/9) + 9n$?

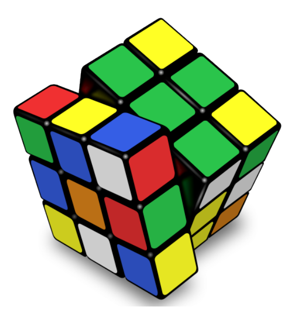
   (h) What is the worst-case running time of quicksort?

   (i) Let $X[1 .. n, 1 .. n]$ be a fixed array of numbers. Consider the following recursive function:

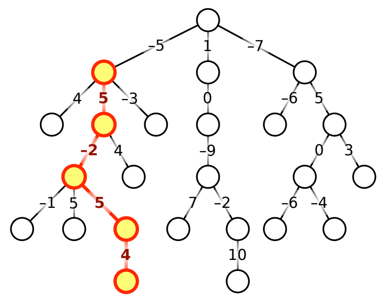
   $$WTF(i,j) = \begin{cases} 0 & \text{if } \min\{i,j\} \le 0 \\ -\infty & \text{if } \max\{i,j\} > n \\ X[i,j] + \max\left\{ \begin{array}{l} WTF(i-2, j+1) \\ WTF(i-2, j-1) \\ WTF(i-1, j-2) \\ WTF(i+1, j-2) \end{array} \right\} & \text{otherwise} \end{cases}$$

   How long does it take to compute $WTF(n,n)$ using dynamic programming?

   (j) The Rubik's Cube is a mechanical puzzle invented in 1974 by Ernő Rubik, a Hungarian professor of architecture. The puzzle consists of a $3 \times 3 \times 3$ grid of 'cubelets', whose faces are covered with stickers in six different colors. In the puzzle's solved state, each face of the puzzle is one solid color. A mechanism inside the puzzle allows any face of the cube to be freely turned (as shown on the right). The puzzle can be scrambled by repeated turns. Given a scrambled Rubik's Cube, how long does it take to find the *shortest* sequence of turns that returns the cube to its solved state?



2. Let T be a rooted tree with integer weights on its edges, which could be positive, negative, or zero. The weight of a path in T is the sum of the weights of its edges. Describe and analyze an algorithm to compute the maximum weight of any path from a node in T up to one of its ancestors. It is not necessary to compute the actual maximum-weight path; just its weight. For example, given the tree shown below, your algorithm should return the number 12.



The maximum-weight upward path in this tree has weight 12.

3. Describe and analyze efficient algorithms to solve the following problems:

   (a) Given a set of n integers, does it contain two elements a, b such that $a + b = 0$?

   (b) Given a set of n integers, does it contain three elements a, b, c such that $a + b + c = 0$?

4. Describe and analyze an efficient algorithm to find the length of the longest substring that appears both forward and backward in an input string $T[1\,..\,n]$. The forward and backward substrings must not overlap. Here are several examples:

   - Given the input string `ALGORITHM`, your algorithm should return 0.
   - Given the input string `RECURSION`, your algorithm should return 1, for the substring `R`.
   - Given the input string `REDIVIDE`, your algorithm should return 3, for the substring `EDI`. (The forward and backward substrings must not overlap!)
   - Given the input string `DYNAMICPROGRAMMINGGETSMANYPOINTS`, your algorithm should return 4, for the substring `YNAM`.

5. *[Taken directly from HBS0.]* Recall that the *Fibonacci numbers* F_n are recursively defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for every integer $n \geq 2$. The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$.

   Prove that any non-negative integer can be written as the sum of distinct *non-consecutive* Fibonacci numbers. That is, if any Fibonacci number F_n appears in the sum, then its neighbors F_{n-1} and F_{n+1} do not. For example:

$$\begin{aligned}
88 &= 55 + 21 + 8 + 3 + 1 &&= F_{10} + F_8 + F_6 + F_4 + F_2 \\
42 &= 34 + 8 &&= F_9 + F_6 \\
17 &= 13 + 3 + 1 &&= F_7 + F_4 + F_2
\end{aligned}$$