This exam lasts 180 minutes.
Write your answers in the separate answer booklet.
Please return this question sheet with your answers.

1. A boolean formula is in *disjunctive normal form* (or *DNF*) if it consists of a *disjunction* (OR) or several *terms*, each of which is the conjunction (AND) of one or more literals. For example, the formula

$$(\overline{x} \wedge y \wedge \overline{z}) \vee (y \wedge z) \vee (x \wedge \overline{y} \wedge \overline{z})$$

is in disjunctive normal form. DNF-SAT asks, given a boolean formula in disjunctive normal form, whether that formula is satisfiable.

   (a) Describe a polynomial-time algorithm to solve DNF-SAT.

   (b) What is the error in the following argument that P=NP?

> *Suppose we are given a boolean formula in conjunctive normal form with at most three literals per clause, and we want to know if it is satisfiable. We can use the distributive law to construct an equivalent formula in disjunctive normal form. For example,*
>
> $$(x \vee y \vee \overline{z}) \wedge (\overline{x} \vee \overline{y}) \iff (x \wedge \overline{y}) \vee (y \wedge \overline{x}) \vee (\overline{z} \wedge \overline{x}) \vee (\overline{z} \wedge \overline{y})$$
>
> *Now we can use the algorithm from part (a) to determine, in polynomial time, whether the resulting DNF formula is satisfiable. We have just solved 3SAT in polynomial time. Since 3SAT is NP-hard, we must conclude that P=NP!*

2. The University of Southern North Dakota at Hoople has hired you to write an algorithm to schedule their final exams. Each semester, USNDH offers n different classes. There are r different rooms on campus and t different time slots in which exams can be offered. You are given two arrays $E[1..n]$ and $S[1..r]$, where $E[i]$ is the number of students enrolled in the ith class, and $S[j]$ is the number of seats in the jth room. At most one final exam can be held in each room during each time slot. Class i can hold its final exam in room j only if $E[i] < S[j]$.

   Describe and analyze an efficient algorithm to assign a room and a time slot to each class (or report correctly that no such assignment is possible).

3. A *shuffle* of two strings X and Y is formed by interspersing the characters into a new string, keeping the characters of X and Y in the same order. For example, '`bananaananas`' is a shuffle of '`banana`' and '`ananas`' in several different ways.

$$\texttt{banana}_{\texttt{ananas}} \qquad \texttt{ban}_{\texttt{ana}}\texttt{ana}_{\texttt{nas}} \qquad \texttt{b}_{\texttt{an}}\texttt{an}_{\texttt{a}}\texttt{a}_{\texttt{na}}\texttt{na}_{\texttt{s}}$$

   The strings '`prodgyrnamammiincg`' and '`dyprongarmammicing`' are both shuffles of '`dynamic`' and '`programming`':

$$\texttt{pro}^{\texttt{d}}\texttt{g}^{\texttt{y}}\texttt{r}^{\texttt{nam}}\texttt{ammi}^{\texttt{i}}\texttt{n}^{\texttt{c}}\texttt{g} \qquad \texttt{dy}_{\texttt{pro}}\texttt{n}^{\texttt{a}}\texttt{g}^{\texttt{m}}\texttt{r}_{\texttt{amm}}\texttt{ic}_{\texttt{ing}}$$

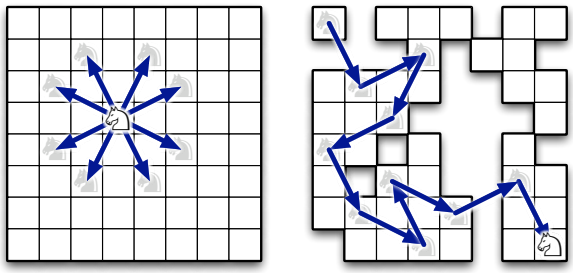
   Given three strings $A[1..m]$, $B[1..n]$, and $C[1..m+n]$, describe and analyze an algorithm to determine whether C is a shuffle of A and B.

4. Suppose we are given an array $A[1..n]$ with the special property that $A[1] \geq A[2]$ and $A[n-1] \leq A[n]$. We say that an element $A[x]$ is a *local minimum* if it is less than or equal to both its neighbors, or more formally, if $A[x-1] \geq A[x]$ and $A[x] \leq A[x+1]$. For example, there are six local minima in the following array:

9	7	7	2	1	3	7	5	4	7	3	3	4	8	6	9

We can obviously find a local minimum in $O(n)$ time by scanning through the array. Describe and analyze an algorithm that finds a local minimum in $O(\log n)$ time. *[Hint: With the given boundary conditions, the array* ***must*** *have at least one local minimum. Why?]*

5. A *knight* is a chess piece that can move either two squares horizontally and one square vertically, or two squares vertically and one square horizontally, in a single turn. Suppose you are given an $n \times n$ chessboard with some squares removed. You want to move a knight from the upper left corner of this partial chessboard to the lower right corner without (1) landing on a removed square or (2) moving in exactly the same direction twice in a row. Describe an efficient algorithm that either computes the shortest sequence of knight moves that accomplishes this task, or correctly reports that the task is impossible. *[Hint: Build a graph.]*



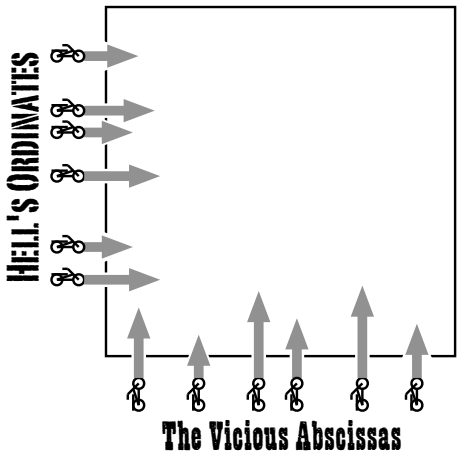
Left: All possible moves for a knight on an 8×8 chessboard.
Right: A legal sequence of knight moves on a partial 8×8 chessboard.

6. Jeff tries to make his students happy. At the beginning of each semester, he passes out a questionnaire that lists a number of possible course policies in areas where he is flexible. Every student is asked to respond to each possible course policy with one of "strongly favor", "mostly neutral", or "strongly oppose". Each student may respond with "strongly favor" or "strongly oppose" to at most five questions. Because Jeff's students are very understanding, each student is happy if (but only if) he or she prevails in just one of his or her strong policy preferences.

   Jeff would like to set his course policies so that the number of happy students is maximized. ***Prove*** that this problem is NP-hard. *[Hint: Reduce from 3SAT.]*

7. It's time for the 3rd Quasi-Annual Champaign-Urbana Ice Motorcycle Demolition Derby Race-O-Rama and Spaghetti Bake-Off! The main event is a competition between two teams of n motorcycles in a huge square ice-covered arena. All of the motorcycles have spiked tires so that they can ride on the ice. Each motorcycle drags a long metal chain behind it. Whenever a motorcycle runs over a chain, the chain gets caught in the tire spikes, and the motorcycle crashes. Two motorcycles can also crash by running directly into each other. All the motorcycle start simultaneously. Each motorcycle travels in a straight line at a constant speed until it either crashes or reaches the opposite wall—no turning, no braking, no speeding up, no slowing down. The Vicious Abscissas start at the south wall of the arena and ride directly north (vertically). Hell's Ordinates start at the west wall of the arena and ride directly east (horizontally). If any motorcycle completely crosses the arena, that rider's entire team wins the competition.

   Describe and analyze an efficient algorithm to decide which team will win, given the starting position and speed of each motorcycle.



You may assume the following problems are NP-hard:

CircuitSat: Given a boolean circuit, are there any input values that make the circuit output True?

PlanarCircuitSat: Given a boolean circuit drawn in the plane so that no two wires cross, are there any input values that make the circuit output True?

3Sat: Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?

MaxIndependentSet: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MaxClique: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MinVertexCover: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

MinDominatingSet: Given an undirected graph G, what is the size of the smallest subset S of vertices such that every vertex in G is either in S or adjacent to a vertex in S?

MinSetCover: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subcollection whose union is S?

MinHittingSet: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subset of S that intersects every subset S_i?

3Color: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

ChromaticNumber: Given an undirected graph G, what is the minimum number of colors needed to color its vertices, so that every edge touches vertices with two different colors?

MaxCut: Given a graph G, what is the size (number of edges) of the largest bipartite subgraph of G?

HamiltonianCycle: Given a graph G, is there a cycle in G that visits every vertex exactly once?

HamiltonianPath: Given a graph G, is there a path in G that visits every vertex exactly once?

TravelingSalesman: Given a graph G with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

SubsetSum: Given a set X of positive integers and an integer k, does X have a subset whose elements sum to k?

Partition: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3Partition: Given a set X of n positive integers, can X be partitioned into $n/3$ three-element subsets, all with the same sum?

Minesweeper: Given a Minesweeper configuration and a particular square x, is it safe to click on x?

Tetris: Given a sequence of N Tetris pieces and a partially filled $n \times k$ board, is it possible to play every piece in the sequence without overflowing the board?

Sudoku: Given an $n \times n$ Sudoku puzzle, does it have a solution?

KenKen: Given an $n \times n$ Ken-Ken puzzle, does it have a solution?