CS473 Algorithms : Lecture 10 (2022-02-18)

logistics : - pset 3 due
- pset 4 out

last lecture : - flows - review
- FF is $\boxed{\text{pseudo}}$ poly
- $\boxed{\text{bad}}$ augmenting paths
- scaling FF in $O(m^2 \lg F)$

today: flows

thm: $G=(V,E)$ capacitated (directed) graph
w/ $\boxed{\text{integral}}$ capacities. $s,t \in V$
(s,t)-max flow can be computed in $O(mF)$
$O(m|f^*|)$
$O(m^2 \lg F)$
...
$O(nm)$ [Orlin 13]

$\boxed{\text{given}}$ (s,t)-max flow, can compute (s,t) min cut in $O(m)$ time

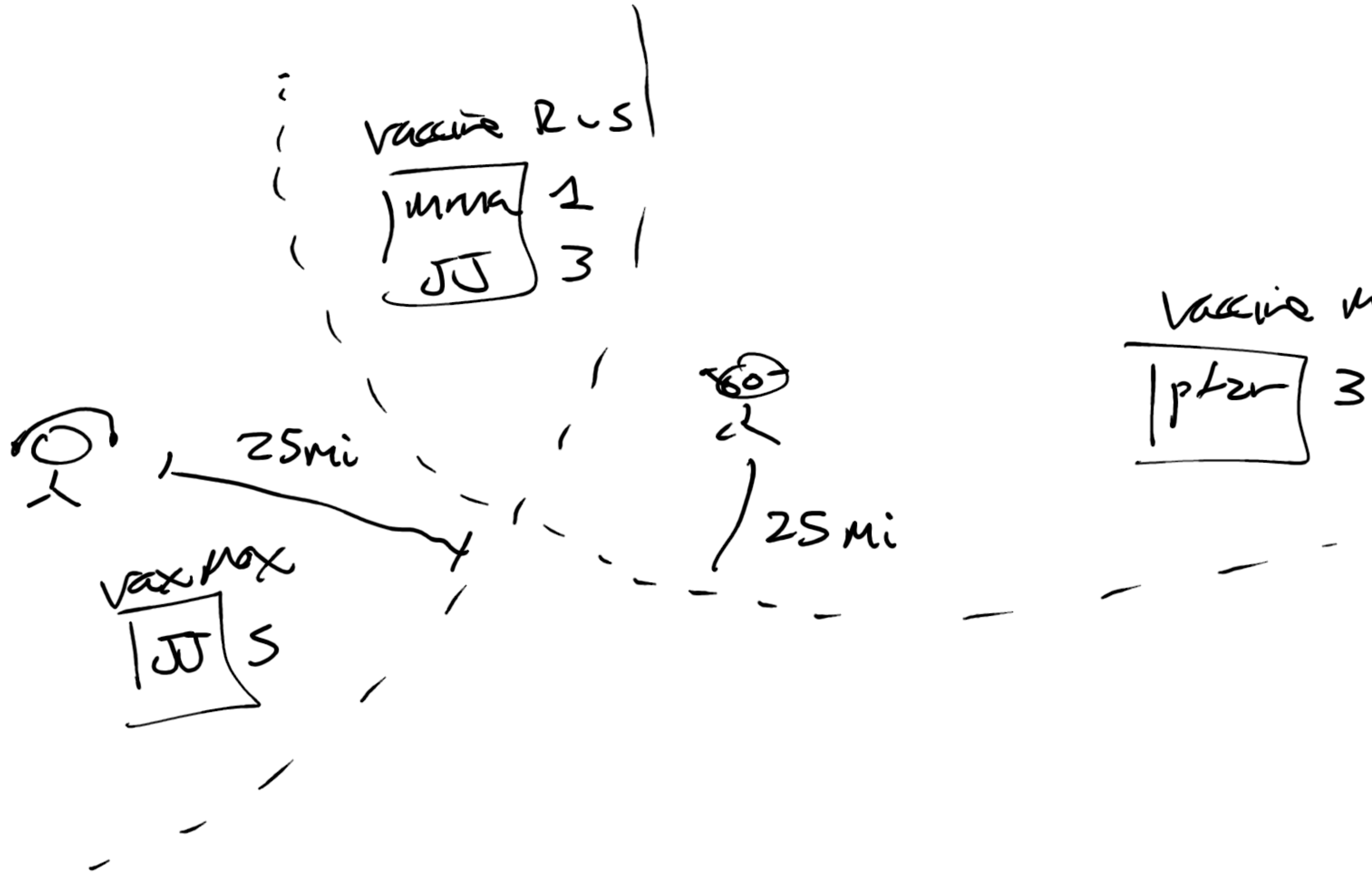
Convention: use when solving max flow in coursework

Q: what $\boxed{\text{other}}$ problems can we solve?

idea: use $\boxed{\text{reductions}}$

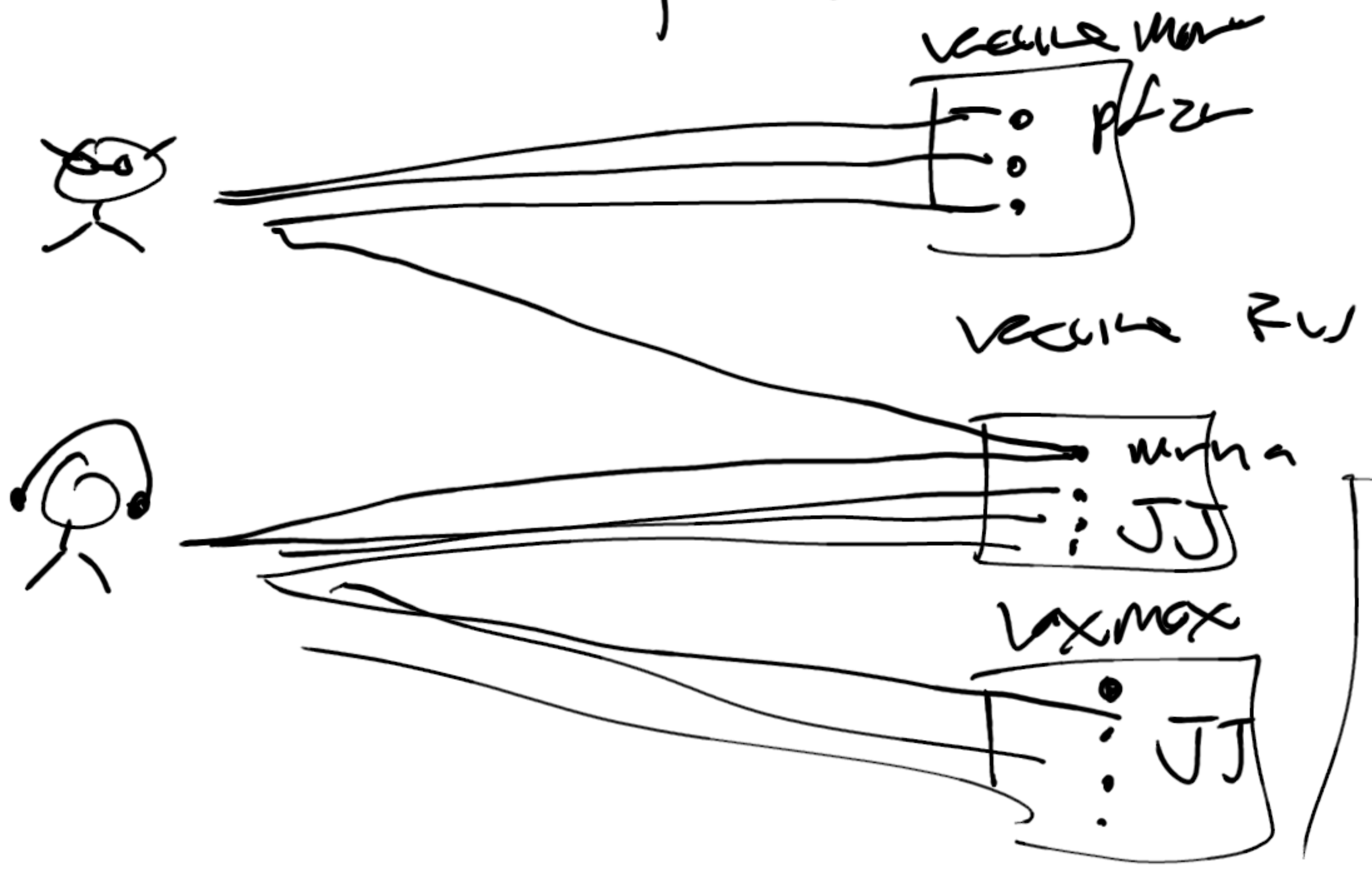
eg: $G=(V,E)$ has property P $\mapsto$ G' has max flow value $\geq k$

Q: can we get everyone vaccinated quickly?



Q: assign each person to acceptable vaccine dose within 25mi?

brand prefs



the bipartite ________

Q: is there a matching in the graph?

def: a bipartite (undirected) graph is an undirected graph $G=(V,E)$

$V = L \cup R$

$E \subseteq L \times R$

def: undirected G, a matching is $M \subseteq E$ st each vertex $v \in V$ is incident to ≤ 1 edge in M

is perfect $\forall v \in V$ ________ $= 1$ edge in M

lem: in bipartite graph G, G has perfect matching only if $|L| = |R|$

def: the maximum matching problem is to compute

$$\arg\max_{M \text{ matching in } G} |M|$$

Q: compute maximum bipartite matching?

idea: develop algo from scratch

idea: solve by $\boxed{\text{reduction}}$ to maxflow

construction: G bipartite graph on $L \cup R$

define directed capacitated graph $G' = (V', E')$

$$V' = \{s\} \cup L \cup R \cup \{t\}$$

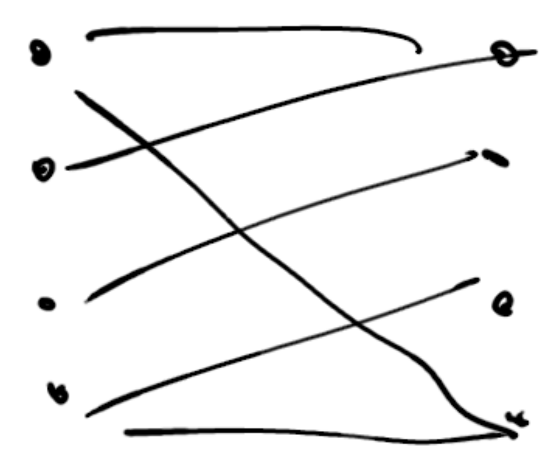
$$E' = E \quad \text{view as } E \subseteq L \times R$$

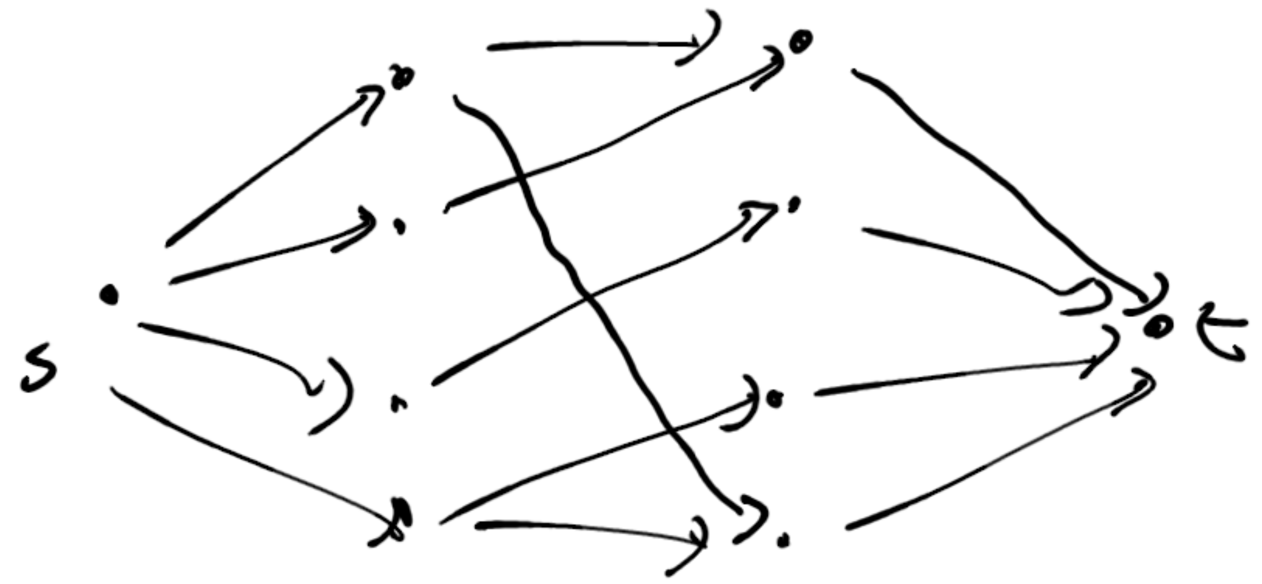
$$\cup \{(s,u) : u \in L\}$$

$$\cup \{(v,t) : v \in R\}$$

$$c'_{e'} = 1 \quad \text{all } e' \in E'$$

eg:

G: 

G': 

all capacities are 1

prop: G has matching $\{(u_{ij}, v_{ij})\}_i$ of size k

$$\underset{L}{u_{ij}} \qquad \underset{R}{v_{ij}}$$

$\Rightarrow$ G' has $\{0,1\}$-valued flow f

$$f_{e'} = \begin{cases} 1 & e' = (u,v) \in M \\ 1 & e' = (s,u) \quad u \in L,\ \exists v \in R\ (u,v) \in M \\ 1 & e' = (v,t) \quad v \in R,\ \exists u \in L\ (u,v) \in M \\ 0 & \text{else} \end{cases}$$

w/ $|f| = k$

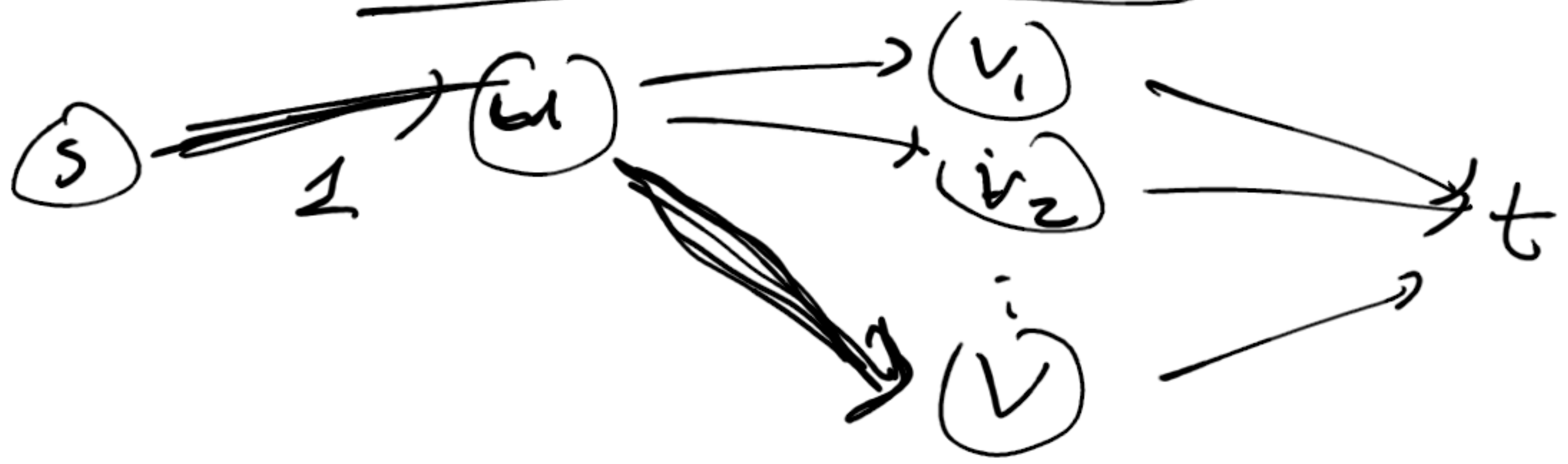
pf: Clm: f is $\{0,1\}$-valued flow

Clm: f valid flow

pf: capacity: flow values are $\{0,1\} \subseteq [0, 1]$, $1 = c_{e'}$ all $e' \in E'$

conservation: s, t: nothing to show

$u \in L$: $\exists v \in R\ (u,v) \in M$:



$$f^{in}(u) = f_{su} = 1$$

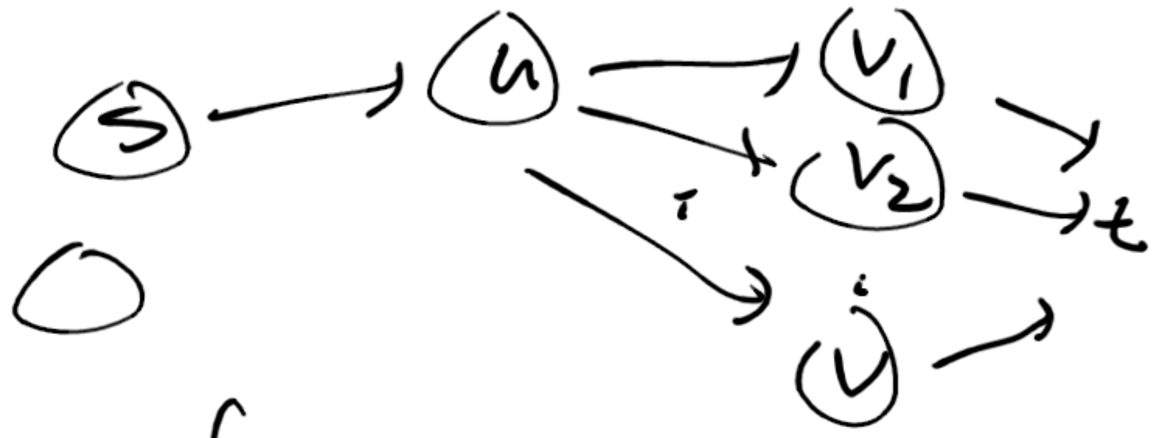
$$f^{out}(u) = \sum_{e' : u \xrightarrow{e'} v \in R} f_{e'} = f_{uv} = 1$$

≤ 1 matched edge

≥ 1 matched edge (u,v)

$$A(u) = \underbrace{f^{out}(u)}_{1} - \underbrace{f^{in}(u)}_{1} = 0$$

$\nexists v \in R\ (u,v) \in M$:



$$f^{in}(u) = f_{su} = 0$$

$$f^{out}(u) = \sum_{e' : u \xrightarrow{e'} v \in R} f_{e'} = 0$$

$f_{e'} = 1$ iff $e' \in M$, $= 0$ else

$$A(u) = f^{out}(u) - f^{in}(u) = 0$$

$v \in R$: analogous

clm: $|f| = k$

pf: $$|f| = f(s) = \sum_{e' : s \xrightarrow{e'} u} \underbrace{f_{e'}}_{= \begin{cases} 1 & \text{if } u \text{ matched in } M \\ 0 & \text{else} \end{cases}}$$

$$= |M| = k$$

$$M = \{ (u_{i_1}, v_{i_1}) \ldots (u_{i_k}, v_{i_k}) \} \subseteq E$$

all u_{i_j} distinct

$\square$

$\Rightarrow$ f valid flow w/ value $|M|$

Prop: G' $\{0,1\}$-valued flow of value $|f|$

$$M = \{ (\underset{\substack{\uparrow \\ L}}{u}, \underset{\substack{\uparrow \\ R}}{v}) : f_{uv} = 1 \} \subseteq E$$

$\Rightarrow$ M is a matching in G size $|f|$

pf: clm: M is a matching

pf. clm: $u \in L$ has ≤ 1 matched edge $(u,v) \in M$

pf. $|\{v \in R : (u,v) \in M\}|$

$= |\{v : (u,v) \in E \;\; f_{uv} = 1\}|$

$= \sum_{e' : u \xrightarrow{e'} v} f_{e'}$ ← f $\{0,1\}$-valued

$= f^{out}(u)$

$= f^{in}(u)$

$= f_{su} \leq c_{su} = 1$

$\square$

clm: $v \in R$ has ≤ 1 matched edge

pf: analogous

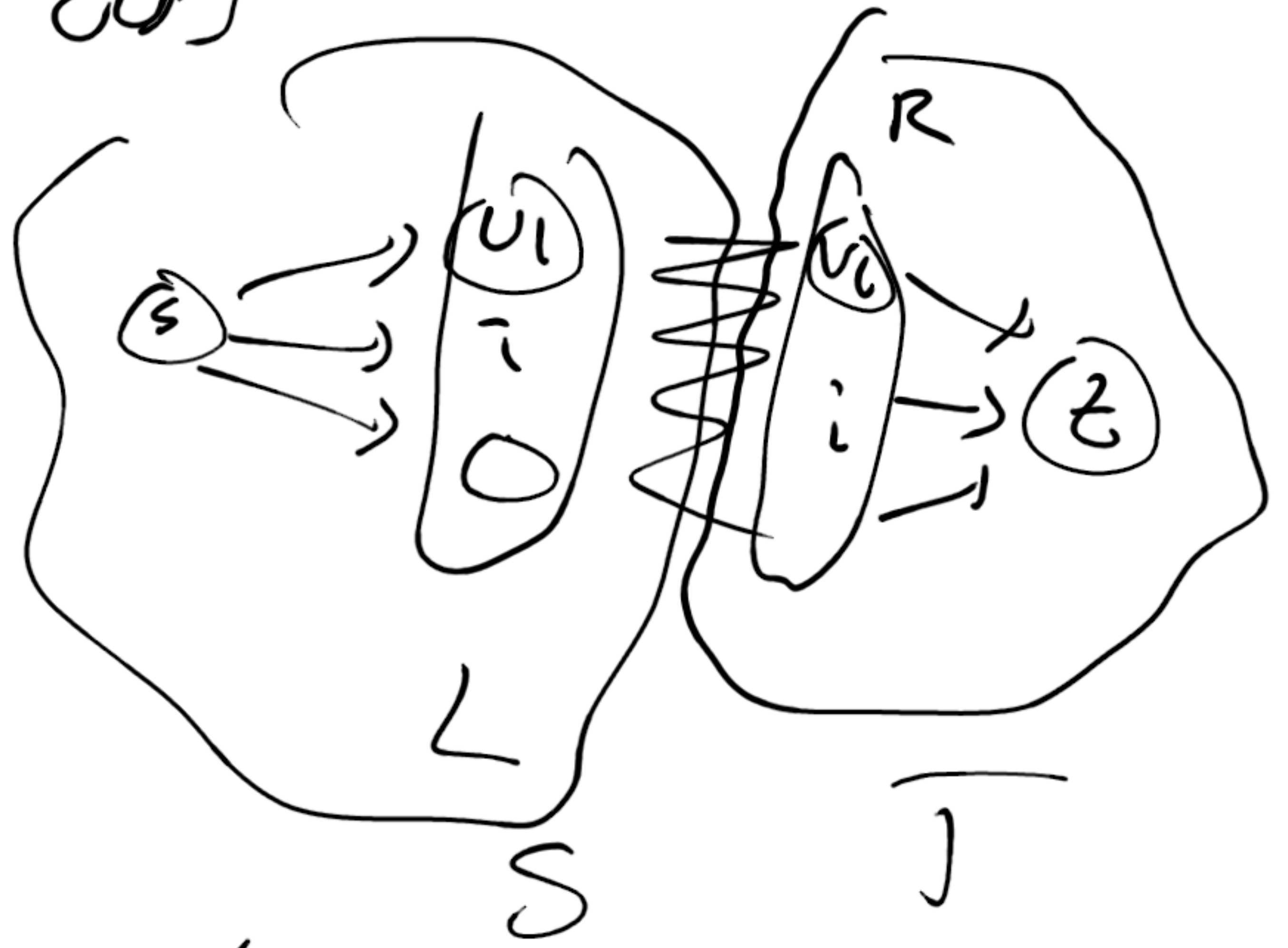
$\Rightarrow$ M matching $\square$

clm: $|M| = |f|$

pf: idea: flows vs cuts

$S = \{s\} \cup L$

$T = R \cup \{t\}$



clm: (S,T) is (s,t)-cut in G'

clm: $|M| = |f|$

pf: $|f| = f(s)$

$= f(S)$

$= \underbrace{f^{out}(S)}\; - \; \underbrace{f^{in}(S)}$

$= \sum_{e' : \underset{\substack{\uparrow \\ S}}{u} \xrightarrow{e'} \underset{\substack{\uparrow \\ T}}{v}} f_{e'}$ $\qquad$ $= \underbrace{\sum_{e' : \underset{\substack{\uparrow \\ S}}{u} \leftarrow \underset{\substack{\uparrow \\ T}}{v}} f_{e'}}_{\text{no edges}} = 0$

$= \sum_{e' : \underset{\substack{\uparrow \\ L}}{u} \to \underset{\substack{\uparrow \\ R}}{v}} \underbrace{f_{e'}} = |M|$

$f_{e'} = \begin{cases} 1 & e' \in M \\ 0 & e' \notin M \end{cases}$

$\Rightarrow |M| = |f|$ $\square$

thm: $\max_{\text{flow } f \text{ in } G'} |f| = \max_{M \text{ matching in } G} |M|$

pf. $\geq$: M matching in $G \mapsto$ flow f in G' value $|M|$

$\leq$: max flow f^* in G value $|f^*|$

$\Rightarrow$ ______________ , f^* integral flow

$\Rightarrow$ ______________ , $\{0,1\}$-values

all flow values $\in [0,1]$

$\Rightarrow$ matching in G size $|f^*|$ $\square$

thm: bipartite maximum matching solvable in $O(nm)$ time

pf: algo: $O(m)$ (1) construct G' from G

$O(m|f^*|)$ (2) compute integral max flow f^* in G'

$O(m)$ (3) derive matching M from f^*

(4) return M

correctness: clear

complexity: $O(m|f^*|)$

claim: $|f^*| \leq n$

pf: $|\hat{C}(\{s\}, V' \setminus \{s\})|$

$= \sum_{e' = s \xrightarrow{e'} \cdot} \underbrace{c_{e'}}_{1}$

$= |L| \leq n$

vert in G

$\square$

$\Rightarrow O(nm)$ runtime

rmk: - natural instance when $|f^*| \leq \text{poly}(n,m)$

- scaling FF would naively take $O(m^2 \log F) \gg O(nm)$

today = flow - bipartite matching
- $\boxed{\text{reduced}}$ bipartite matching to max flow

next lecture.. flows

logistics - pset 3 due
pset 4 out