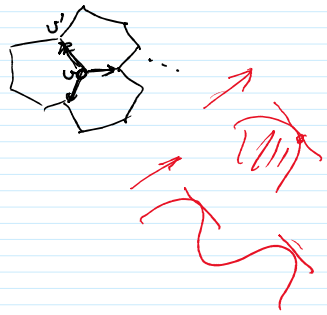
Simplex Method (Dantzig '47)

idea - local improvement again!

v = initial vertex of feasible region
repeat {
  find an adjacent vertex v' of v in feasible region with larger objective value
  if none exists, return v
  v ← v'
}



Terms:

a pt in feasible region ⟷ feasible soln
a vertex in feasible region ⟷ basic soln
↑
n of m+n constraints are tight

finding an adjacent vertex ⟷ finding a pivot to swap in & out of the basis

Implementation Details: by example

Ex

$$\begin{aligned} \max\ & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.}\ & 2x_1 + 3x_2 + x_3 \le 5 \\ & 4x_1 + x_2 + 2x_3 \le 11 \\ & 3x_1 + 4x_2 + 2x_3 \le 8 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

rewrite:

$$\begin{aligned} \max\ z &= 5x_1 + 4x_2 + 3x_3 \\ w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \ge 0 \end{aligned}$$

slack form

$\frac{5}{2}, \frac{11}{4}, \frac{8}{3}$

initial soln: $(\bar{x}_1, \bar{x}_2, \bar{x}_3; \bar{w}_1, \bar{w}_2, \bar{w}_3) = (0,0,0; 5, 11, 8)$
$\bar{z} = 0$

$1\left(2x_1 + \tfrac{3}{2}x_2 + x_3 \le 5\right)$
$+ 1\left(3x_1 + 4x_2 + 2x_3 \le 8\right)$
$\Rightarrow 5x_1 + 7x_2 + 3x_3 \le 13$

Iteration 1: increase $\bar{x}_1$ to 2.5
x_1 enters basis, w_1 leaves

substitute $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$

4 − 1.5·5
3 − 5·0.5
11 − 4·2.5
+4·0.5

rewrite:

$$\begin{aligned} \max\ z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ \text{s.t.}\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \ge 0 \end{aligned}$$

$\frac{2.5}{0.5}$ ← $\frac{0.5}{0.5}$

$w_3 = 0.5 + 1.5w_1 + 0.5x_2$ $\quad 0.5$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

new soln: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (2.5, 0, 0; 0, 1, 0.5)$

$\bar{z} = 12.5$

Iteration 2: increase $\bar{x}_3$ to 1

x_3 enters, w_3 leaves

substitute $x_3 = 1 + 3w_1 + x_2 - 2w_3$

≤ 13

$$\begin{aligned} \max \quad & z = 13 - w_1 - 3x_2 - w_3 \\ \text{s.t.} \quad & x_1 = 2 - 2w_1 - 2x_2 + 0.5w_3 \\ & w_2 = 1 + 2w_1 + 5x_2 \\ & x_3 = 1 + 3w_1 + x_2 - 2w_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

$y_1 = 1$

$y_3 = 1$

$y_2 = 0$

new soln: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (2, 0, 1; 0, 1, 0)$

$\bar{z} = 13$

done!

correctness obvious

Analysis:

each iteration takes $O(mn)$ time

iterations?

trivial upper bd $\leq$ total # vertices / bases (if no degeneracies)

$$= \binom{m+n}{n}$$

$$\leq 2^{m+n}$$ exponential!

Klee-Minty '73: $\exists$ examples which require exponential time for simplex algm under various pivot-choosing strategies

$\sqrt{n}$ 2 rand

Kalai / Matousek-Sharir-Welzl

But: in practice, # iterations tend to be smaller

Duality

How to certify optimality?

e.g. $\max z = 5x_1 + 4x_2 + 3x_3$

$$\begin{cases} 2x_1 + 3x_2 + x_3 \le 5 & \cdot 0.6 \\ 4x_1 + x_2 + 2x_3 \le 11 & \cdot 0.5 \\ 3x_1 + 4x_2 + 2x_3 \le 8 & \cdot 0.7 \end{cases}$$

$x_1, x_2, x_3 \ge 0$

1.2 +2 +2.1

1.8 +0.5 +2.8

0.6 +1 +1.4

$\Rightarrow z \le 5.3x_1 + 5.1x_2 + 3x_3 \le 14.1$

3 5.5 5.6

How to find the best linear combination?

it's another LP!

let y_1, y_2, y_3 be multipliers

$$\min \; 5y_1 + 11y_2 + 8y_3$$

$$\begin{cases} 2y_1 + 4y_2 + 3y_3 \ge 5 \\ 3y_1 + y_2 + 4y_3 \ge 4 \\ y_1 + 2y_2 + 2y_3 \ge 3 \end{cases}$$

$y_1, y_2, y_3 \ge 0$

Def Given primal LP

$$\max \; c_1x_1 + \cdots + c_nx_n$$
$$\text{s.t. } a_{11}x_1 + \cdots + a_{1n}x_n \le b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m$$
$$x_1, \cdots, x_n \ge 0$$

its dual is:

$$\min \; b_1y_1 + \cdots + b_ny_n$$
$$\text{s.t. } a_{11}y_1 + \cdots + a_{m1}y_m \ge c_1$$
$$\vdots$$
$$a_{1n}y_1 + \cdots + a_{mn}y_m \ge c_n$$
$$y_1, \cdots, y_m \ge 0.$$

$y_1, \dots, y_m \geq 0$.

Obs value of any feasible soln to primal LP $\leq$ value of any feasible soln of dual LP

Thm (LP Duality)

max value of primal LP = min value of dual LP.

Pf: ($\leq$) done.

($\geq$). run simplex alg'm.
get multipliers from final slack form.

□

Rmk: max-flow, min-cut thm is a special case.

How to find initial, basic feasible soln (if origin is not feasible)?

add an extra var x_0

e.g. max $x_1 + x_2$
s.t. $2x_1 - x_2 \leq 2$
$x_1 - 5x_2 \leq -4$
$x_1, x_2 \geq 0$

$\Rightarrow$

"auxiliary LP"
max $-x_0$
$-x_0 + 2x_1 - x_2 \leq 2$
$-x_0 + x_1 - 5x_2 \leq -4$
$x_0, x_1, x_2 \geq 0$

initial soln: $(\bar{x_0}, \bar{x_1}, \bar{x_2}) = (4, 0, 0)$

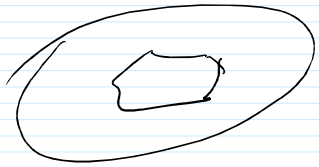
if opt soln to aux LP has $x_0 \neq 0$,
LP is infeasible

Other Methods

Other Methods

Ellipsoid Method (Khachiyan '79)

first polynomial-time algm
but slower than simplex in practice



Interior-Point Methods (Karmarkar '84, ...)
polynomial & more practical ...