Claim. # iterations $\leq mn$

Pf omitted.

$\Rightarrow$ total time $\boxed{O(m^2 n)}$ ←

(best algm: '22: $O(m^{1+\varepsilon} \log U)$)

Rmk - min-cost flow

Given cost: $E \to \mathbb{R}^+$ and d,

find flow f with value $|f| = d$

minimizing $\sum_{(u,v) \in E} cost(u,v) \cdot f(u,v)$

Appl - min-cost perfect matching etc.

Problem (Linear Programming (LP))

maximize / minimize a linear objective fn

s.t. linear inequality constraints over n real vars

i.e. $\max \; c_1 x_1 + \cdots + c_n x_n$

s.t. $a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1$

$\vdots$

$a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m$

"standard form"

$x_1, \ldots, x_n \geq 0$ reals

Ex farmer has 100 acres of land

loan \$60000, 600 units of storage

each acre of corn $\Rightarrow$ cost \$300, 8 units of storage per acre, profit \$800

" " soybean $\Rightarrow$ cost \$800, 2 units of storage per acre, profit \$900

x_1 = #acres for corn, x_2 = soybean

$\max \; 800 x_1 + 900 x_2$

s.t. $x_1 + x_2 \leq 100$

$300 x_1 + 800 x_2 \leq 60000$

$8 x_1 + 2 x_2 \leq 600$

$$8x_1 + 2x_2 \le 600$$
$$x_1, x_2 \ge 0.$$

Appl'n / Special Case: max flow

$$\max \sum_{(s,v)\in E} x_{s,v}$$

$$\text{s.t. } 0 \le x_{u,v} \le c(u,v) \quad \forall (u,v)\in E$$

$$\sum_{w:(u,v)\in E} x_{u,v} - \sum_{w:(v,w)\in E} x_{v,w} = 0 \quad \forall v \in V$$

(note: each var appears in 2 constraints
one with coeff +1
one with coeff −1)

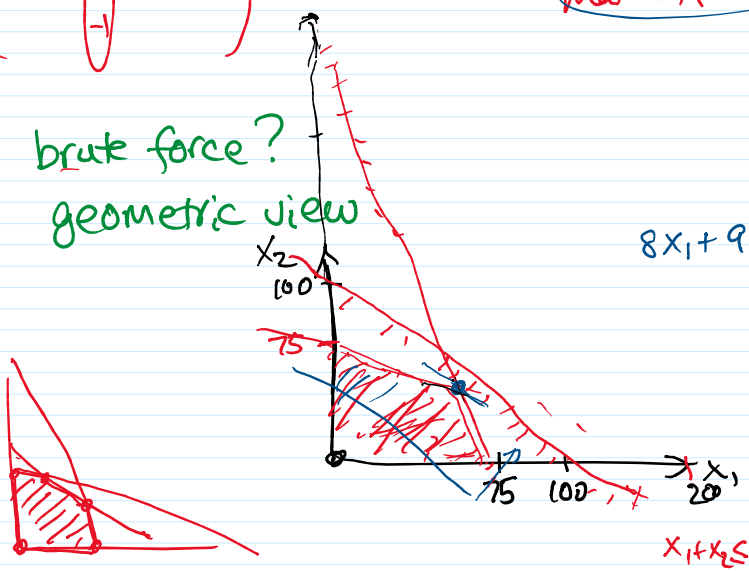
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\max 8x_1 + 9x_2$

$3x_1 + 8x_2 \le 600$

brute force?

geometric view



$8x_1 + 9x_2 = c$

$8x_1 + 2x_2 \le 600$

$x_1 + x_2 \le 100$

$\frac{200 \cdot 3}{8}$

feasible region is a convex polygon for $n=2$
or convex polyhedron in general

goal: find extreme vertex of this polyhedron along a given direction.

Simplex Method (Dantzig '47)

idea - local improvement again!

v = initial vertex of feasible region

repeat {

find an adjacent vertex v' of v in feasible region with larger objective value

if none exists, return v

$v \leftarrow v'$

}



Terms:

a pt in feasible region $\iff$ feasible soln

a vertex in feasible region $\iff$ basic soln

↑ n of $m+n$ constraints are tight ←

finding an adjacent vertex $\iff$ finding a pivot to swap in & out of the basis

Implementation Details: by example

Ex max $5x_1 + 4x_2 + 3x_3$

s.t. $2x_1 + 3x_2 + x_3 \le 5$

$4x_1 + x_2 + 2x_3 \le 11$

$3x_1 + 4x_2 + 2x_3 \le 8$

$x_1, x_2, x_3 \ge 0$

rewrite: max $z = 5x_1 + 4x_2 + 3x_3$

$w_1 = 5 - 2x_1 - 3x_2 - x_3$

$w_2 = 11 - 4x_1 - x_2 - 2x_3$

$w_3 = 8 - 3x_1 - 4x_2 - 2x_3$

slack form

$x_1, x_2, x_3, w_1, w_2, w_3 \ge 0$

$\frac{5}{2}, \frac{11}{4}, \frac{8}{3}$

initial soln: $(\bar{x}_1, \bar{x}_2, \bar{x}_3; \bar{w}_1, \bar{w}_2, \bar{w}_3) = (0,0,0; 5, 11, 8)$

$\bar{z} = 0$

Iteration 1: increase $\bar{x}_1$ to 2.5

x_1 enters basis, w_1 leaves

$4 - 1.5 \cdot 5$

$3 - 5 \cdot 0.5$

Iteration 1: increase x_1 to 2.5

x_1 enters basis, w_1 leaves

$3 - 5 \cdot 0.5$

$11 - 4 \cdot 2.5$

$+4 \cdot 0.5$

substitute $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$

rewrite: max $z = 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3$

$x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$

$w_2 = 1 + 2w_1 + 5x_2$

$w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

new sol'n: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (2.5, 0, 0; 0, 1, 0.5)$

$\bar{z} = 12.5$