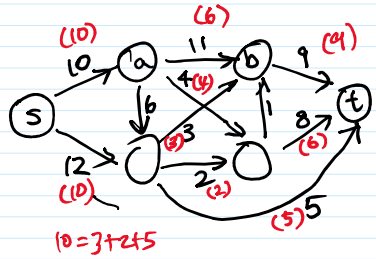
**Problem** Given directed graph (network) $G=(V,E)$, $|V|=n$, $|E|=m$, $s,t \in V$

($s$: source (in-deg 0), $t$: sink (out-deg 0))

where each $e \in E$ has a capacity $c(e) > 0$,

find flow from $s$ to $t$ with max value




value 20

**Def** A flow is $f: E \to \mathbb{R}$ s.t.

$\forall (u,v) \in E, \quad 0 \le f(u,v) \le c(u,v)$

flow conservation: $\forall v \in V - \{s,t\}, \quad \sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w)$

Value of $f$: $|f| = \sum_{v:(s,v)\in E} f(s,v)$

$\left( = \sum_{v:(v,t)\in E} f(v,t) \right)$.

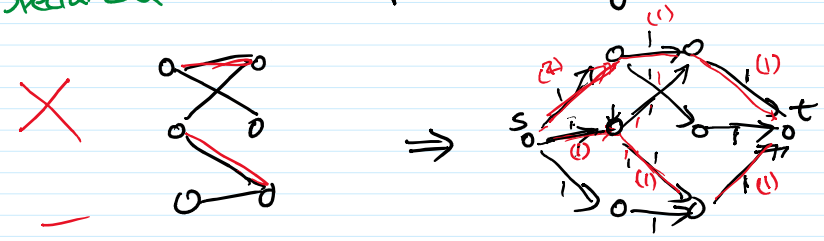
**Convention:** make $f(u,v) = -f(v,u)$ if $(v,u) \in E$

& $f(u,v) = 0$ if $(u,v), (v,u) \notin E$

flow conservation becomes $\forall v \in V-\{s,t\} \sum_{w \in V} f(v,w) = 0$.

**Appl'ns**

**Special case 1:** max bipartite matching




(turns out: opt flow is integral)

**Special case 2:** assignment problems

jobs ↔ people

each person $u$ can be $\le c_u$ jobs

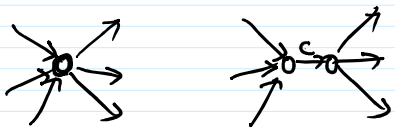
⋮

each person ... can be ... 1

⋮

**Special case 3:** find max # of edge-disjoint paths from s to t in dir. graphs

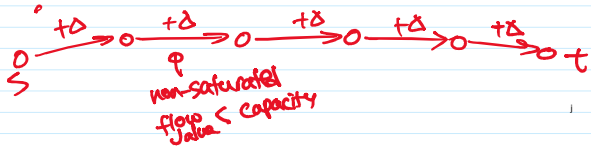
(can also do dir. vertex-disjoint)

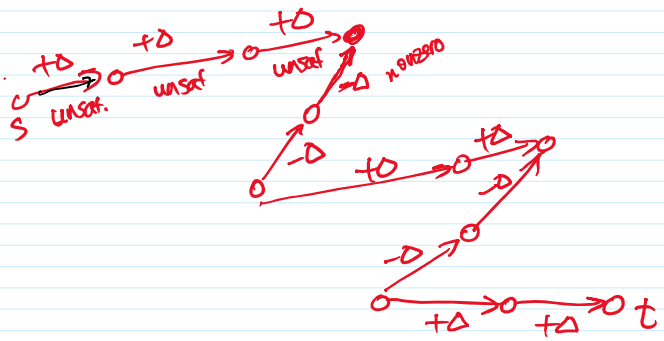


etc.

## Ford-Fulkerson Alg'm ('56)

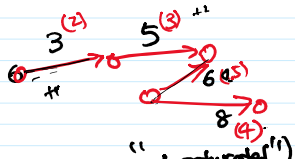
idea - local improvement again




**Def** Let f be a flow.

Define residual graph $G_f$:




if $(u,v) \in E$, $f(u,v) < c(u,v)$ ("not saturated")

add $(u,v)$ in $G_f$ with capacity $c_f(u,v) = c(u,v) - f(u,v)$

if $(v,u) \in E$, $f(v,u) > 0$ ("not zero")

add $(u,v)$ in $G_f$ with capacity $c_f(u,v) = f(v,u)$

**Alg'm:**

$f \equiv$ zero flow

repeat {

→ find an s-to-t path p in $G_f$ ← called augmenting path

if not exist, return f

$\Delta =$ min edge capacity in p in $G_f$ ← called bottleneck value

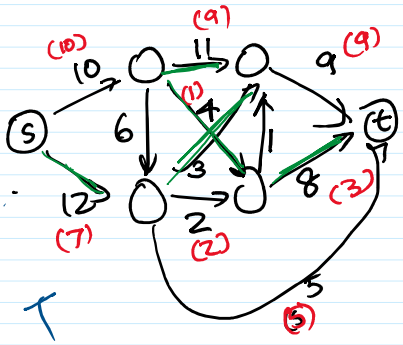
$\Delta$ = min edge capacity in $p$ in $G_f$

↖ called bottleneck value

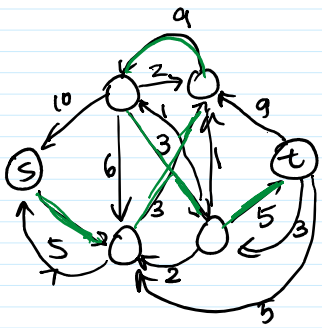
$\forall (u,v) \in p$, $f(u,v) \leftarrow f(u,v) + \Delta$

$f(v,u) \leftarrow f(v,u) - \Delta$.

Ex

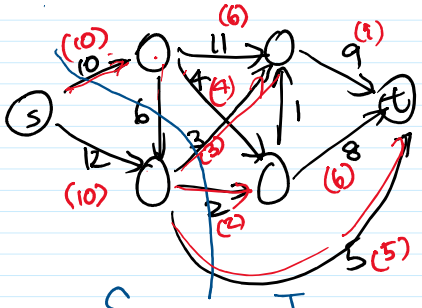



f:



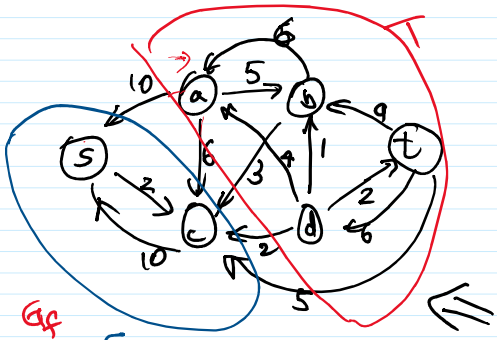

$G_f$:

set $\Delta = 3$




S T

flow value 10+10=20




$G_f$

S

no path

done

{s, c}

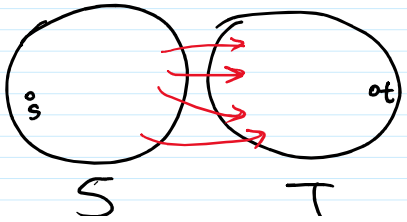
## Correctness:

**Thm** flow $f$ is optimal iff it has no augmenting path.

How to certify optimality?

**Def** An s-t cut is a partition $(S,T)$ of $V$ with $s \in S$, $t \in T$

The capacity of $(S,T) = \sum_{u \in S, v \in T, (u,v) \in E} c(u,v)$






**Obs** value of any flow $\leq$ capacity of any $(s,t)$-cut

**Pf:**

$$|f| = \sum_{u \in S, v \in T} f(u,v) \leq \sum_{\substack{u \in S, v \in T \\ (u,v) \in E}} c(u,v) - \cancel{\sum_{\substack{u \in S, v \in T \\ (v,u) \in E}} 0}$$

$$= \text{capacity of } (S,T). \quad \square$$

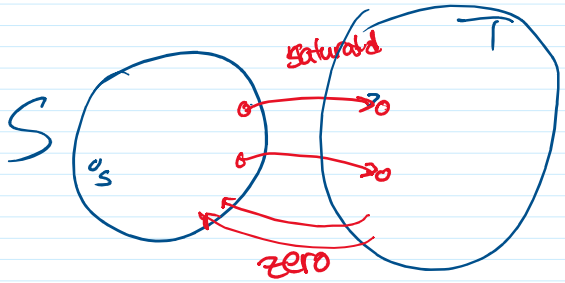
**Thm 2** value of max flow = Capacity of min $(s,t)$-cut

## Pf of Both Thms:

Let $f$ be flow with no aug path.

Let $S$ = all vertices reachable from $s$ in $G_f$

$T = V - S$ $\quad t \in T$




$\forall u \in S, v \in T, \quad (u,v) \in E \Rightarrow f(u,v) = c(u,v)$

$(v,u) \in E \Rightarrow f(u,v) = 0$

$$\text{max flow value} \geq |f| = \sum_{u \in S, v \in T} f(u,v) = \sum_{\substack{u \in S, v \in T \\ (u,v) \in E}} c(u,v) = \text{capacity of } (S,T) \geq \text{capacity of min cut} \geq \text{max flow value}$$

$\square$

## Runtime Analysis:

## Runtime Analysis:

each iteration takes $O(m)$ time

if capacities are integers in $\{1, \dots, U\}$,

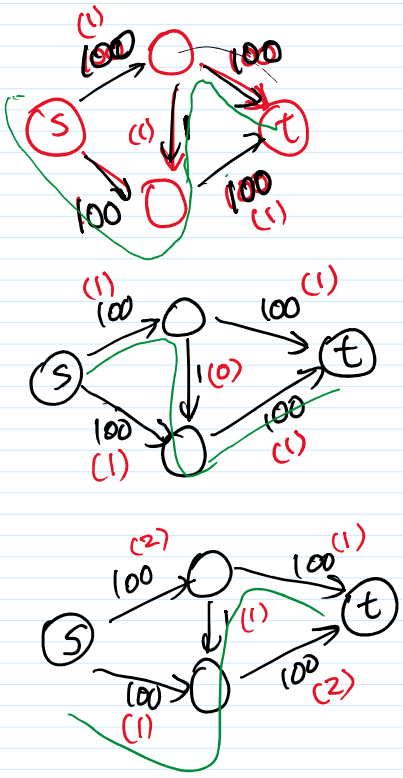
$$\#\text{ iterations} \leq |f^*| \leq nU$$

($f^*$: opt flow)

$$\Rightarrow \boxed{O(mnU)} \text{ total time}$$

not strongly polynomial

## Bad Ex:




If non-integers, may not even terminate!

## Cor (Integrality Thm)

If all capacities are integers,
opt flow has integer values.

---

## Edmonds-Karp Version 1 ('72)

always pick aug path with largest bottleneck value

always pick aug path with largest bottleneck value

(by using variant of Dijkstra) in $O(m \log n)$ time

Analysis:

Claim # iterations $\leq O(m \log |f^*|)$

Pf Sketch: let $f_i^*$ be max flow in $G_f$ in $i^{th}$ iteration

Fact (Flow Decomposition Thm)
any flow can be expressed as a sum of $\leq m$ path flows & $\leq m$ cycle flows.

decompose into $\leq m$ path flows & $\leq m$ cycle flows

$\Rightarrow \exists$ path flow with $\geq \frac{|f_i^*|}{m}$ value

augmenting path

$\Rightarrow |f_{i+1}^*| \leq |f_i^*| - \frac{|f_i^*|}{m}$

$\leq \left(\frac{m-1}{m}\right) |f_i^*|.$

$\Rightarrow$ # iterations $\leq \log_{\frac{m}{m-1}} |f^*|$

$= O\left(\frac{\log |f^*|}{\log\left(1+\frac{1}{m-1}\right)}\right)$

$\leq O(m \log |f^*|)$. □

$\log(1+x) = \Theta(x)$

$\Rightarrow$ total time $O(m^2 \log n \log |f^*|)$

$\leq \boxed{O(m^2 \log^2 (nU))}.$

Edmonds-Karp Version 2:

always pick shortest aug path
(# edges)

each iteration $O(m)$ time

Claim. # iterations $\leq mn$

Pf omitted.

**Claim.** # iterations $\leq mn$

Pf omitted.

$\Rightarrow$ total time $\boxed{O(m^2 n)}$

(best algm: '22: $O(m^{1+\varepsilon} \log U)$ )