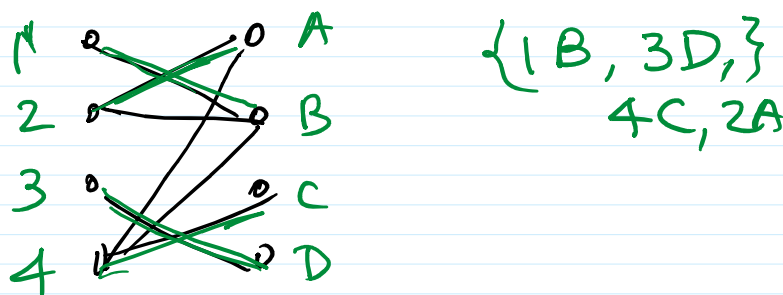


OPTIMIZATION : Matching, Flows, & LP

Problem Given bipartite graph $G = (V, E)$,
find matching with max size
 ↗ subgraph with max deg 1

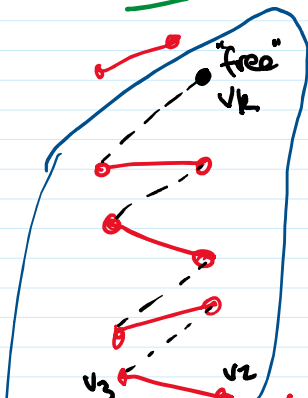


idea - brute force $\Omega(n!)$
 greedy not work
 DP not work ...

"Hungarian" Method (König-Egerváry '55 / Edmonds '65)

idea - local improvement
 (start with feasible sol'n,
 modify & improve, ...)

Def



Let M be a matching.

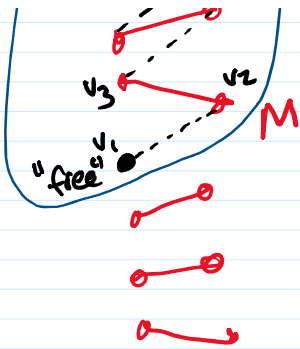
An augmenting path P for M
 (alternating)

is a path $v_1 v_2 v_3 \dots v_k$ in G

st. $v_1 v_2 \notin M$, $v_2 v_3 \in M$,
 $v_3 v_4 \notin M$, $v_4 v_5 \in M$,

$v_{k-1} v_k \notin M$

... .. are not covered by M



$v_k, v_k \notin M$
and v_1, v_k are not covered by M
i.e. are "free"

$\Rightarrow M \oplus P$ is a matching
with $|M \oplus P| \geq |M| + 1$.
($S \oplus T = S \cup T - (S \cap T)$
symmetric diff)

Alg'm:

$M = \phi$. $\leftarrow$
repeat {
 find an augmenting path P for M
 if not exist, return M $\leftarrow$
 $M = M \oplus P$
}

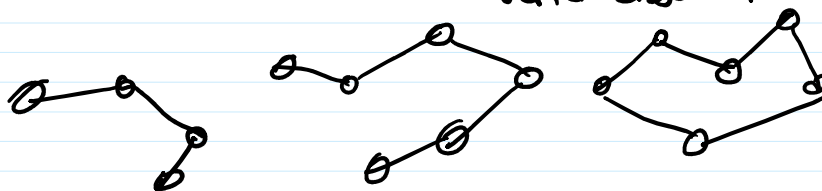
Analysis: # iterations $\leq |M^*| \leq n/2$ $\swarrow$ opt sol'n
runtime $O(n \cdot \text{cost of finding aug path})$

Correctness:

Lemma If M and M^* are matchings
with $|M| < |M^*|$,
then $\exists \geq |M^*| - |M|$ vertex-disjoint
aug. paths for M . $\leftarrow$

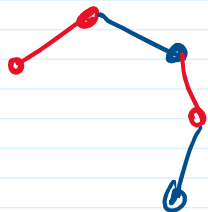
Pf:

Fact any undir. graph with max deg ≤ 2
can be decomposed into
vertex-disjoint paths & cycles

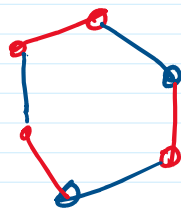


$M \oplus M^*$ has $\text{max deg} \leq 2$.

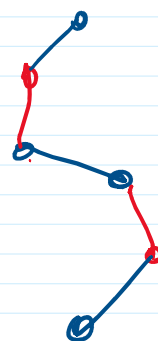
$M \oplus M^*$ has $\max \deg \leq 2$.



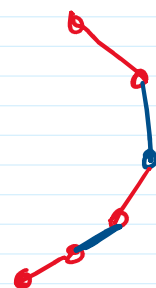
path
of type 0
diff. 0



cycle
of even length
diff. 0



path
type 1
diff 1



path
type 2
diff -1

$$\begin{aligned}
 0 < \underline{|M^*| - |M|} &= \# \text{ paths of type 1} \\
 &\quad - \# \text{ paths of type 2} \\
 &\leq \# \text{ paths of type 1} \\
 &\leq \# \text{ aug paths} \\
 &\quad \text{vertex deg.}
 \end{aligned}$$

□

Thm

A matching M is optimal
 $\Leftrightarrow$ it has no aug. path.

Pf:

if $\exists$ aug path, not opt.

if not opt, $\exists$ aug path by Lemma. □

How to find an augmenting path:

form directed graph G_M : u on left, v on right

if $uv \in E - M$, add (u, v) to G_M

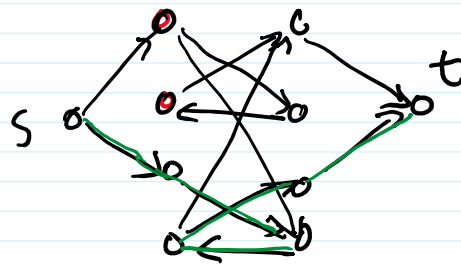
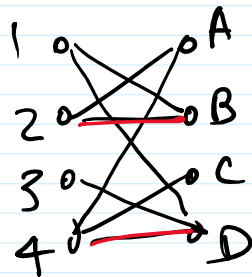
if $uv \in M$, add (v, u) to G_M

if u free, add (s, u) to G_M

if v free, add (v, t) to G_M

if a tree, add (s, u) to M
 if u free, add (u, t) to M

Ex



$$M = \{2B, 4D\}$$

$$3D + C$$

$$\Rightarrow O(m) \text{ time per iteration}$$

$$m = |E| \quad m \geq n$$

$$n = |V|$$

$$\Rightarrow \boxed{O(mn)} \text{ time}$$

Hopcroft & Karp Algm ('73) : Sketch

idea 1 - always pick shortest augmenting path

idea 2 - augment multiple ₁ paths per iteration
 vertex-disjoint $\leftarrow$ (still $O(m)$ time per iteration)

$\Downarrow$

Fact length of shortest aug. paths increases per iteration.

After $\geq \sqrt{n}$ iterations,

$$\text{path lengths} \geq \sqrt{n}$$

$$\Rightarrow \leq \sqrt{n} \text{ vertex-disjoint aug paths}$$

by Lemma $\Rightarrow$

$$|M^*| - |M| \leq \sqrt{n}$$

by Lemma
 $\Rightarrow$

$$|M^*| - |M| \leq \sqrt{n}$$

$$\text{i.e. } |M| \geq |M^*| - \sqrt{n}$$

$$\Rightarrow \# \text{ remaining iterations} \leq \sqrt{n}$$

$$\Rightarrow \boxed{O(m\sqrt{n})} \text{ time}$$

$$\text{van den Brand, Lee, ... '20: } \underline{O(m^{1/2}\epsilon)}$$

Extensions:

max unweighted non-bipartite
Edmonds '65 $O(mn^2)$

$\downarrow$
'80 $O(n^3\sqrt{n})$

max weighted bipartite

" " non-bipartite

min cost perfect matching

⋮