precision issues ...

Appl'n 1. multiplying 2 large n-bit numbers $\left(\sum_{i=0}^{n-1} a_i 2^i\right) \cdot \left(\sum_{i=0}^{n-1} b_i 2^i\right)$

elementary-school method $O(n^2)$

⋮

FFT $O(n \log n)$ ops on $(\log n)$-bit numbers

Schönhage-Strassen '71 $O(n \log n \log\log n)$ bit ops

Fürer '07 $O(n \log n \log\log\log\log\cdots\log n)$ "

Harvey & van der Hoeven '19 $O(n \log n)$.

Appl'n 2. pattern matching with "don't cares"

given 2 strings $a_1 a_2 \cdots a_m \in (\Sigma \cup \{?\})^*$ "pattern"

$b_1 b_2 \cdots\cdots b_n \in \Sigma^*$ "text"

$(m<n)$ decide whether $\exists k = 0..n-m$ s.t. $\underbrace{a_1 \cdots a_m}_{\substack{\forall i=1,..,m,\ a_i = b_{k+i} \\ \text{or } a_i = '?'}} = \underline{b_{k+1} \cdots b_{k+m}}$

e.g. "i??u"

"algorithmisfun"

trivial alg'm: $O(mn)$ time

without "don't care": $O(n)$ time [many known alg'ms]

with "don't care":

Fischer-Paterson '74 $O(n \log n \log |\Sigma|)$ by FFT

Indyk '98
Kalai '02
Cole-Hariharan '02 } $O(n \log n)$ rand.

Simpler Sol'n by Clifford-Clifford '07:

let $z_j = \begin{cases} 0 & \text{if } a_j \text{ is "don't care"} \\ 1 & \text{else} \end{cases}$

define $c_k = \sum_{j=1}^{m} z_j (a_j - b_{k+j})^2 \quad \forall k = 0,..,n-m$

$c_k = \sum_{j=1} z_j \ldots_{k+j}$

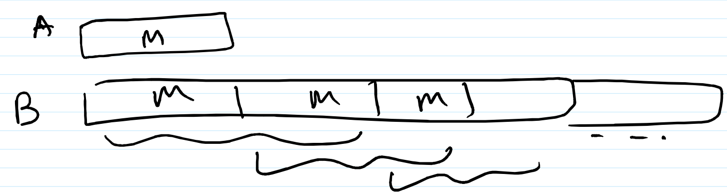
ans is "yes" iff $c_k = 0$ for some k

$$c_k = \underbrace{\sum_{j=1}^{m} z_j a_j^2}_{\text{compute once}} - 2 \underbrace{\sum_{j=1}^{m} \underbrace{z_j a_j}_{A_j} \underbrace{b_{k+j}}_{B_{k+j}}}_{\text{1 convolution}} + \underbrace{\sum_{j=1}^{m} \underbrace{z_j}_{A'_j} \underbrace{b_{k+j}^2}_{B'_{k+j}}}_{\text{1 convolutions}}$$

$$\underbrace{\sum_{j=1}^{m} A_j B_{k+j}}_{\text{convolution after reversing } B\text{'s}}$$

$\Rightarrow \boxed{O(n \log n)}$ time

can be reduced to $O(n \log m)$



$$O\left(\frac{n}{m}(m \log m)\right) = O(n \log m).$$

Matrix Multiplication

Given $n \times n$ matrices A, B, compute $C = AB$

$1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8$

$1 \cdot 3 + 2 \cdot 6 + 3 \cdot 9$

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 36 & 42 & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$$

obvious alg'm (by def'n):

for $i = 1$ to n
  for $j = 1$ to n
    $c_{ij} = \sum^{n} a_{ik} b_{kj}$

$\boxed{O(n^3)}$ time

for $j = 1$ to n

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$\boxed{O(n^3)}$ time

$\Omega(n^2)$

Strassen's Alg'm ('69)

divide $A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ $\quad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$

where A_{ij}, B_{ij} are $\frac{n}{2} \times \frac{n}{2}$

1st idea -

$$AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right)$$

$$\Rightarrow T(n) = 8\,T\left(\frac{n}{2}\right) + O(n^2)$$

$O(1)$ # additions & copying

$$\Rightarrow O(n^{\log_2 8}) = \boxed{O(n^3)}$$

not better?

more clever idea -

$C_1 = A_{11}(B_{11} - B_{21})$ $\quad C_2 = (A_{11} + A_{12})B_{21}$

$C_3 = A_{22}(B_{22} - B_{12})$ $\quad C_4 = (A_{21} + A_{22})B_{12}$

$C_5 = (A_{11} + A_{22})(B_{21} + B_{12})$

$C_6 = (A_{12} - A_{22})(B_{21} + B_{22})$

$C_7 = (A_{21} - A_{11})(B_{11} + B_{12})$

$$AB = \left(\begin{array}{c|c} C_1 + C_2 & C_5 + C_6 + C_3 - C_2 \\ \hline C_5 + C_7 + C_1 - C_4 & C_3 + C_4 \end{array}\right)$$

$$\Rightarrow T(n) = 7\,T\left(\frac{n}{2}\right) + O(n^2)$$

$$\Rightarrow O(n^{\log_2 7}) \leq \boxed{O(n^{2.81})}$$

Better?

3-way D&C: $T(n) = 23\,T\left(\frac{n}{3}\right) + O(n^2)$

$\Rightarrow O(n^{\log_3 23}) \leq O(n^{2.85})$

⋮

Pan '78: $T(n) = \overset{\frac{k^3-4k}{3}+6k^2}{143640}\,T\left(\overset{\frac{n}{k}}{\frac{n}{70}}\right) + O(n^2)$

$\Rightarrow O(n^{\log_{70} 143640}) \leq O(n^{2.796})$

Pan '79: $O(n^{2.781})$

Bini et al. '80: $O(n^{2.780})$

Schönhage '81: $O(n^{2.522})$

Strassen '86: $O(n^{2.479})$

Coppersmith–Winograd '90: $O(n^{2.376})$

Strother '10 $O(n^{2.374})$

Vassilevska W. '12 $O(n^{2.373})$

⋮

$n^{2.3716}$

Dupont et al. Aug'26 $n^{2.371177}$

$2 \leq \omega < 2.371177$