

Some Useful Inequalities

Suppose X is the sum of random indicator variables X_1, X_2, \dots, X_n .
For each index i , let $p_i = \Pr[X_i = 1] = E[X_i]$, and let $\mu = \sum_i p_i = E[X]$.

- **Markov's Inequality:**

$$\Pr[X \geq x] \leq \frac{\mu}{x} \quad \text{for all } x > 0, \text{ and therefore...}$$

$$\Pr[X \geq (1 + \delta)\mu] \leq \frac{1}{1 + \delta} \quad \text{for all } \delta > 0$$

- **Chebyshev's Inequality:** If the variables X_i are pairwise independent, then...

$$\Pr[(X - \mu)^2 \geq z] < \frac{\mu}{z} \quad \text{for all } z > 0, \text{ and therefore...}$$

$$\Pr[X \geq (1 + \delta)\mu] < \frac{1}{\delta^2 \mu} \quad \text{for all } \delta > 0$$

$$\Pr[X \leq (1 - \delta)\mu] < \frac{1}{\delta^2 \mu} \quad \text{for all } \delta > 0$$

- **Higher Moment Inequalities:** If the variables X_i are $2k$ -wise independent, then...

$$\Pr[(X - \mu)^{2k} \geq z] = O\left(\frac{\mu^k}{z}\right) \quad \text{for all } z > 0, \text{ and therefore...}$$

$$\Pr[X \geq (1 + \delta)\mu] = O\left(\frac{1}{\delta^{2k} \mu^k}\right) \quad \text{for all } \delta > 0$$

$$\Pr[X \leq (1 - \delta)\mu] = O\left(\frac{1}{\delta^{2k} \mu^k}\right) \quad \text{for all } \delta > 0$$

- **Chernoff's Inequality:** If the variables X_i are fully independent, then...

$$\Pr[X \geq x] \leq e^{x - \mu} \left(\frac{\mu}{x}\right)^x \quad \text{for all } x \geq \mu, \text{ and therefore...}$$

$$\Pr[X \geq (1 + \delta)\mu] \leq e^{-\delta^2 \mu / 3} \quad \text{for all } 0 < \delta < 1$$

$$\Pr[X \leq (1 - \delta)\mu] \leq e^{-\delta^2 \mu / 2} \quad \text{for all } 0 < \delta < 1$$

- **The World's Most Useful Inequality:** $1 + x \leq e^x$ for all x

- **The World's Most Useful Limit:** $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

Hashing Properties

\mathcal{H} is a set of functions from some universe \mathcal{U} to $[m] = \{0, 1, 2, \dots, m-1\}$.

- **Universal:** $\Pr_{h \in \mathcal{H}} [h(x) = h(y)] \leq \frac{1}{m}$ for all distinct items $x \neq y$
- **Near-universal:** $\Pr_{h \in \mathcal{H}} [h(x) = h(y)] \leq O\left(\frac{1}{m}\right)$ for all distinct items $x \neq y$
- **Strongly universal:** $\Pr_{h \in \mathcal{H}} [h(x) = i \text{ and } h(y) = j] = \frac{1}{m^2}$ for all distinct $x \neq y$ and all i and j
- **2-uniform:** Same as strongly universal.
- **Ideal Random:** Fiction.

1. Recall that a family \mathcal{H} of hash functions is **universal** if $\Pr_{h \in \mathcal{H}}[h(x) = h(y)] \leq 1/m$ for all distinct items $x \neq y$, where m is the size of the hash table. For any fixed hash function h , a **collision** is an **unordered** pair of distinct items $x \neq y$ such that $h(x) = h(y)$.

Suppose we hash a set of n items into a table of size $m = 2n$, using a hash function h chosen uniformly at random from some universal family. Assume \sqrt{n} is an integer.

- (a) **Prove** that the expected number of collisions is at most $n/4$.
- (b) **Prove** that the probability that there are **at least $n/2$ collisions** is **at most $1/2$** .
- (c) **Prove** that the probability that any subset of **more than \sqrt{n} items** all hash to the same address is at most $1/2$. [Hint: Use part (b).]
- (d) Now suppose we choose h at random from a **4-uniform** family of hash functions, which means for all distinct items w, x, y, z and all addresses i, j, k, l , we have

$$\Pr_{h \in \mathcal{H}} [h(w) = i \wedge h(x) = j \wedge h(y) = k \wedge h(z) = l] = \frac{1}{m^4}.$$

Prove that the probability that any subset of more than \sqrt{n} items all hash to the same address is at most $O(1/n)$.

[Hint: All four statements have short elementary proofs via tail inequalities.]

(a) $E\{\#\text{collisions}\} = \sum_{x \neq y} \Pr[x \text{ and } y \text{ collide}] = \sum_{x \neq y} \Pr[h(x) = h(y)]$
def collide
 $\leq \sum_{x \neq y} \frac{1}{2n} = \binom{n}{2} \frac{1}{2n} = \frac{n(n-1)}{2} \cdot \frac{1}{2n}$
 $= \frac{n-1}{4} \leq \frac{n}{4} \checkmark$
universal

(b) $\Pr\{\#\text{collisions} \geq \frac{n}{2}\} \leq \Pr\{\#\text{coll} \geq 2 E\{\#\text{coll}\}\}$
part (a)
 $\stackrel{\text{Markov}}{\leq} \frac{n/4}{n/2} = \frac{1}{2} \checkmark$

(c) $\Pr\{> \sqrt{n} \text{ at same address}\} \leq \Pr\{\#\text{collisions} \geq \binom{\sqrt{n}+1}{2}\}$
 $= \Pr\{\#\text{coll} \geq \frac{\sqrt{n}(\sqrt{n}+1)}{2}\}$
 $\leq \Pr\{\#\text{coll} \geq \frac{n}{2}\} \leq 1/2$

(d) ~~$\Pr\{\#\text{coll} \geq \frac{n}{2}\} \leq \Pr\{\#\text{coll} \geq 2 E\{\#\text{coll}\}\}$~~
 ~~$\stackrel{\text{Cheb.}}{\leq} \frac{1}{E\{\#\text{coll}\}} \rightarrow \frac{4}{n}$~~
#collisions is sum of pairwise ind. ind vars

④ Try again

$$E[\#\{h(x)=0\}] = n \cdot \Pr[h(x)=0]$$
$$= \frac{1}{2} = \boxed{\frac{1}{2}}$$

$$\Pr[\#\{h(x)=0\} > \sqrt{n}] = \Pr[X > [2\sqrt{n} \cdot \mu]]$$

Cheb

$$\leq \frac{1}{8^2 n} = \frac{1}{(2\sqrt{n})^2 \cdot \frac{1}{2}} = \frac{1}{2n}$$

ChebCheb

$$\leq O\left(\frac{1}{8^4 n^2}\right) = O\left(\frac{1}{n^2}\right)$$

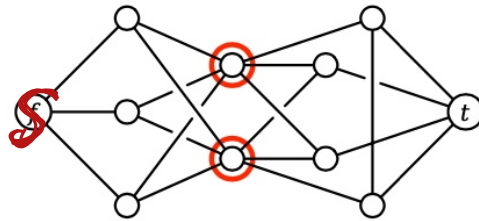
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$$\Pr[\sqrt{n} \text{ in any bucket}] < \frac{1}{2}$$
$$\leq O\left(\frac{1}{n}\right) \quad \checkmark$$

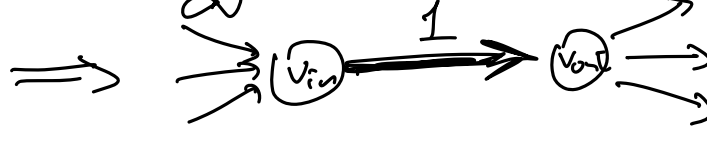
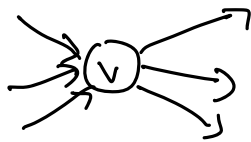
2. The Island of Sodor is home to an extensive rail network. Recently, several cases of a deadly contagious disease have been reported in the village of Ffarquhar. The controller of the Sodor railway plans to close certain railway stations to prevent the disease from spreading to Tidmouth, his home town. No trains can pass through a closed station. To minimize expense (and public notice), he wants to close as few stations as possible. However, he doesn't want to close the Ffarquhar station, because that would expose him to the disease, and he *really* doesn't want to close the Tidmouth station, because then he couldn't visit his favorite pub in Tidmouth.

The Sodor rail network is represented by an undirected graph, with a vertex for each station and an edge for each rail connection between two stations. Two special vertices f and t represent the stations in Ffarquhar and Tidmouth. Describe and analyze an algorithm to find the minimum number of stations *other than f and t* that must be closed to block all rail travel from Ffarquhar to Tidmouth.

For example, given the following input graph, your algorithm should return the integer 2.



min cut



min cut in modified graph
 = min # black edges
 separating s and t

$O(VE)$
 FF = min # vertices ...

3. Let T be a treap with n vertices.

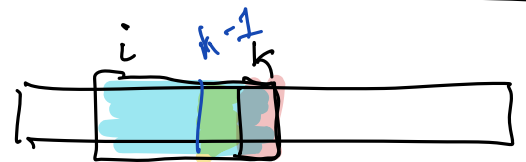
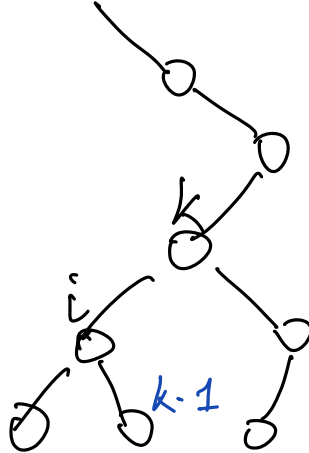
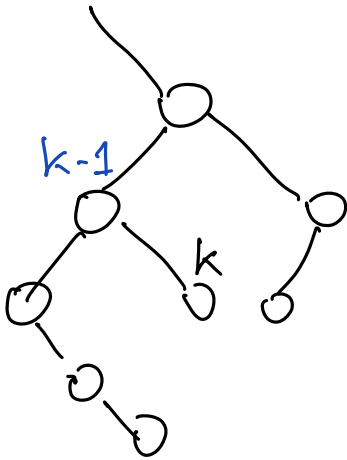
Assume keys $1 \dots n$

- (a) What is the *exact* expected number of leaves in T ?
- (b) What is the *exact* expected number of nodes in T that have two children?
- (c) What is the *exact* expected number of nodes in T that have exactly one child?

You do not need to prove that your answers are correct. [Hint: What is the probability that the node with the k th smallest search key has no children, one child, or two children?]

$$(a) E[\# \text{leaves}] = \sum_{k=1}^n \Pr[k \text{ is a leaf}]$$

$$\Pr[i \uparrow k] = \Pr[i \text{ has smallest priority among } \{i \dots k\}]$$



k has left child i
 $\Rightarrow k$ has smallest priority in $\{i \dots k\}$
 $\Rightarrow \text{priority}(k) < \text{priority}(k-1)$

$$\Pr[k \text{ is a leaf}] =$$

$$\Pr[\text{pr}(k) > \text{pr}(k-1) \text{ and } \text{pr}(k) > \text{pr}(k+1)]$$

~~$\frac{1}{4}$~~

$$= \Pr[\text{priority}(k) = \max \text{ priority } \{k-1, k, k+1\}]$$

$$= \frac{1}{3} \text{ if } k \neq 1 \text{ and } k \neq n$$

$$\frac{n-2}{3} + \frac{2}{2} = \frac{n+1}{3} \text{ if } n > 1$$

$$= \frac{1}{2} \text{ if } k=1 \text{ or } n$$

$$= 1 \text{ if } k=n=1$$

$$(b) E[\# \text{two kids}] = \sum_k \Pr[k \text{ has 2 kids}]$$

$$= \sum_k \Pr[\text{priority}(k) \text{ is } \min \text{ pri } \{k-1, k, k+1\}]$$

$$= \frac{n-2}{3}$$


$$(c) n - \frac{n+1}{3} - \frac{n-2}{3} = \frac{n+1}{3}$$

4. A cyclic shift of a string $A[1..n]$ is any string formed from A by moving a prefix of A to the end, or equivalently, moving a suffix of A to the beginning. For example, For example, the strings $RA!ABRACADAB$ and $DABRA!ABRACA$ and $ABRACADABRA!$ are all cyclic shifts of the string $ABRACADABRA!$.

- (a) Describe and analyze an algorithm to determine, given two strings $A[1..n]$ and $B[1..n]$, whether A is a cyclic shift of B .
- (b) Describe a fast algorithm to determine, given two strings $A[1..m]$ and $B[1..n]$ with $m \leq n$, whether A is a substring of some cyclic shift of B .



(a) Is B a substring of AA ?
 Is $m = n$ and \uparrow
 KMP $O(n)$

(b) 
 Is A a substring of BB ?
 KMP $O(n)$

(c) Is any cyclic shift of A
 a substring of B ?

Hint: prefixes + suffixes separately

- Run KMP, record value of j at every index i
 length of longest prefix of A ending at $B[i]$
- Same for $rev(A)$, $rev(B) \rightarrow$ suffixes