Max flows + min cuts

Input "flow network"
- directed graph $G=(V,E)$
- two vertices s and t
- capacities $c(u\to v)>0$

Maximum flow

Compute $f: E\to\mathbb{R}$
- $f(u\to v)\geq 0$
- $f(u\to v)\leq c(u\to v)$
- $\sum_u f(u\to v) = \sum_w f(v\to w)$ $\quad v\neq s,t$

maximize $|f| = \sum_w f(s\to w) - \sum_u f(u\to s)$

$= \sum_u f(u\to t) - \sum_w f(t\to w)$

Minimum cut

Compute partition $V = S\sqcup T$ $\quad S\cap T=\emptyset$
$s\in S \quad t\in T$

minimize $\|S,T\| = \sum_{u\in S}\sum_{v\in T} c(u\to v)$

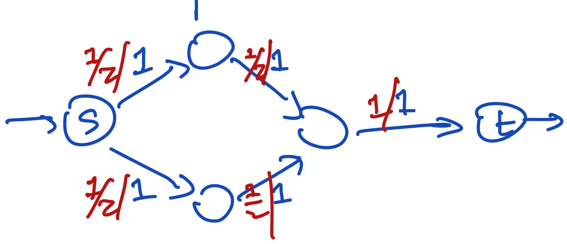
$$\boxed{\max_f |f| = \min_{S,T} \|S,T\|}$$

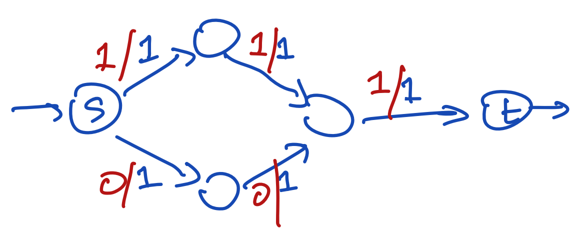
Ford-Fulkerson(G,s,t,c):

```
f ← 0
G_f ← G
while there is a path from s to t in G_f     O(E)
    augment f along that path   } O(V)
    recompute G_f               }
return f
```

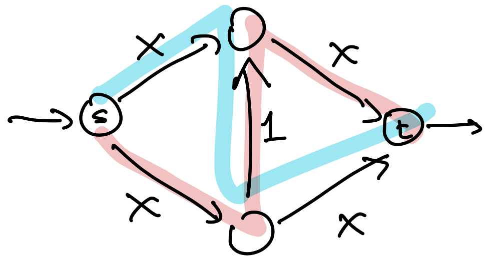
$\boxed{\text{IF}}$ this halts, returns max flow

IF all capacities are integers, max flow is also integral.





$\Rightarrow$ FF runs in $O(E \cdot |F^*|)$ time

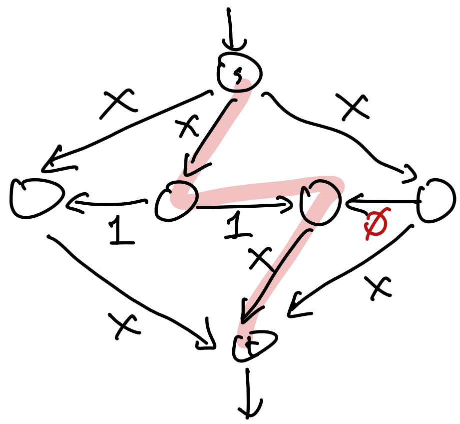


$x \geq 10$

If choose bad paths

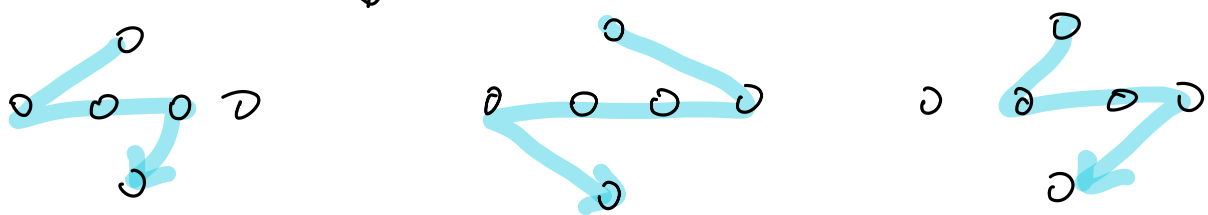
$\Rightarrow$ # iterations $= |F^*| = 2X$.

If we allow irrational capacities



$X \geq 10$

$\phi = \frac{\sqrt{5}-1}{2} = 0.6181\ldots$



A B C

$\downarrow$ B C B A B C B A B C B A ...

1 ϕ ϕ ϕ^2 ϕ^2 ϕ^3 ϕ^3 ....

total flow $= 1 + \sum_{i \geq 1} \phi^i$

$< 7. \ll 2X+1$!

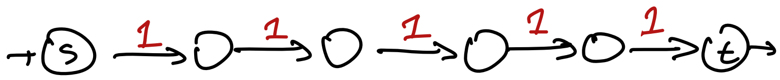
Flow = assignment of numbers to edges satisfying constraints

= sum of paths and cycles

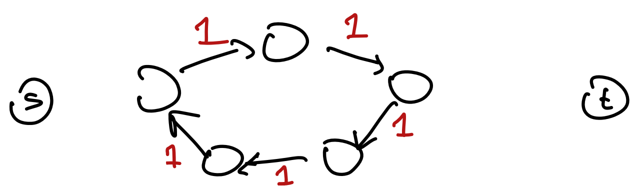
Two flows F and F' $\Rightarrow$ F+F' is also a flow

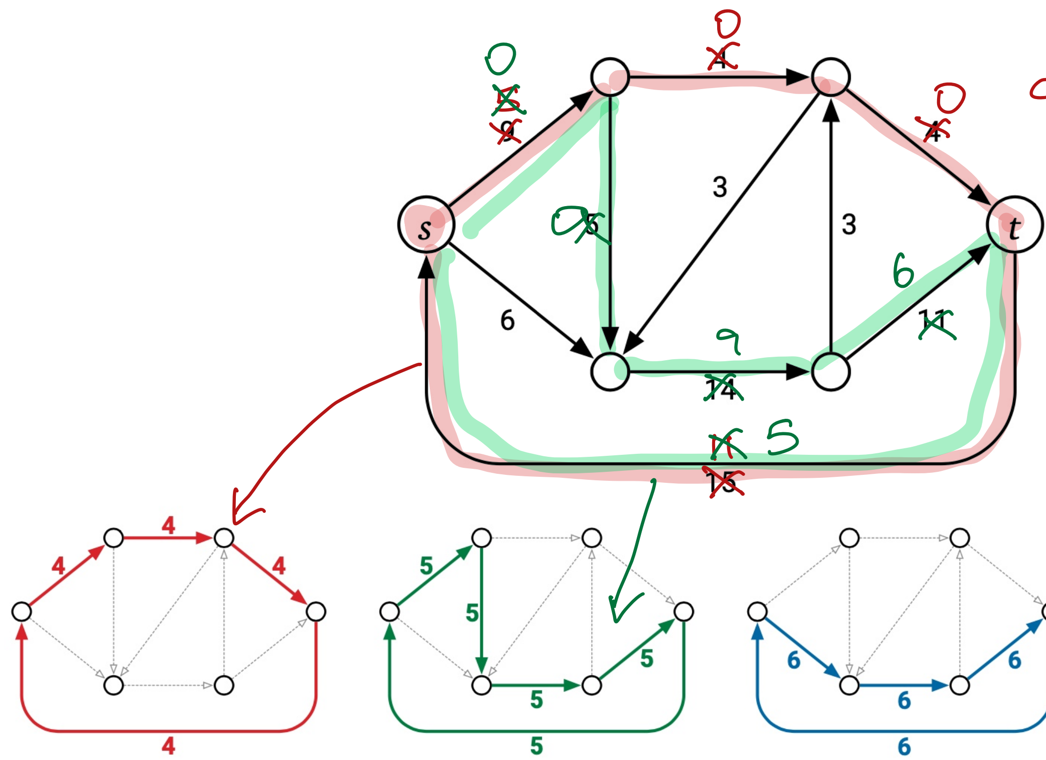
kF is also a flow for any constant k

Paths are flows



Cycles are flows





circulation $\equiv$ flow with $|f|=0$

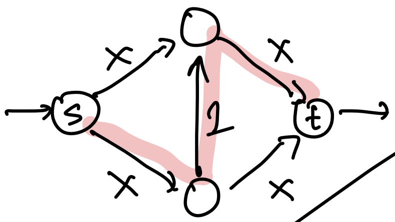
<u>Decompose Circulation(f):</u>

if $f \neq 0$
- Find any cycle C — $O(V)$ time
- $f_{min} = \min_{e \in C} f(e)$
- $f \leftarrow f - f_{min} \cdot C$

Done after $\leq E$ iterations — $\boxed{O(VE) \text{ time}}$

1. Any flow is weighted sum of paths and cycles that only use <u>forward</u> edges
2. Acyclic flow $\longrightarrow$ sum of paths
3. There is an acyclic max flow
4. Representation of flow has complexity $O(VE)$

   There are flows where every decomp. has complexity $\Omega(VE)$



① Fattest path

② Shortest path $=$ min #edges

Edmonds-Karp maximize min capacity

best-first search $O(E \log V)$ time

#iterations $\leq E \cdot \ln|f^*|$

~~Some~~ Fattest path from s to t carries $\geq |f^*|/E$ flow

After k iteration

residual graph has $\leq \left(1 - \frac{1}{E}\right)^k \cdot |f^*|$

flow left to find

WMU# $\boxed{(1+x) \leq e^x}$

Shortest aug. path $O(E)$ per iteration

#iters $= O(VE)$

$O(VE^2)$ time

Orlin 2012: $O(VE)$ time

upper bound on int caps.

Chen et al April 2022: $O(E^{1+o(1)} \log U)$