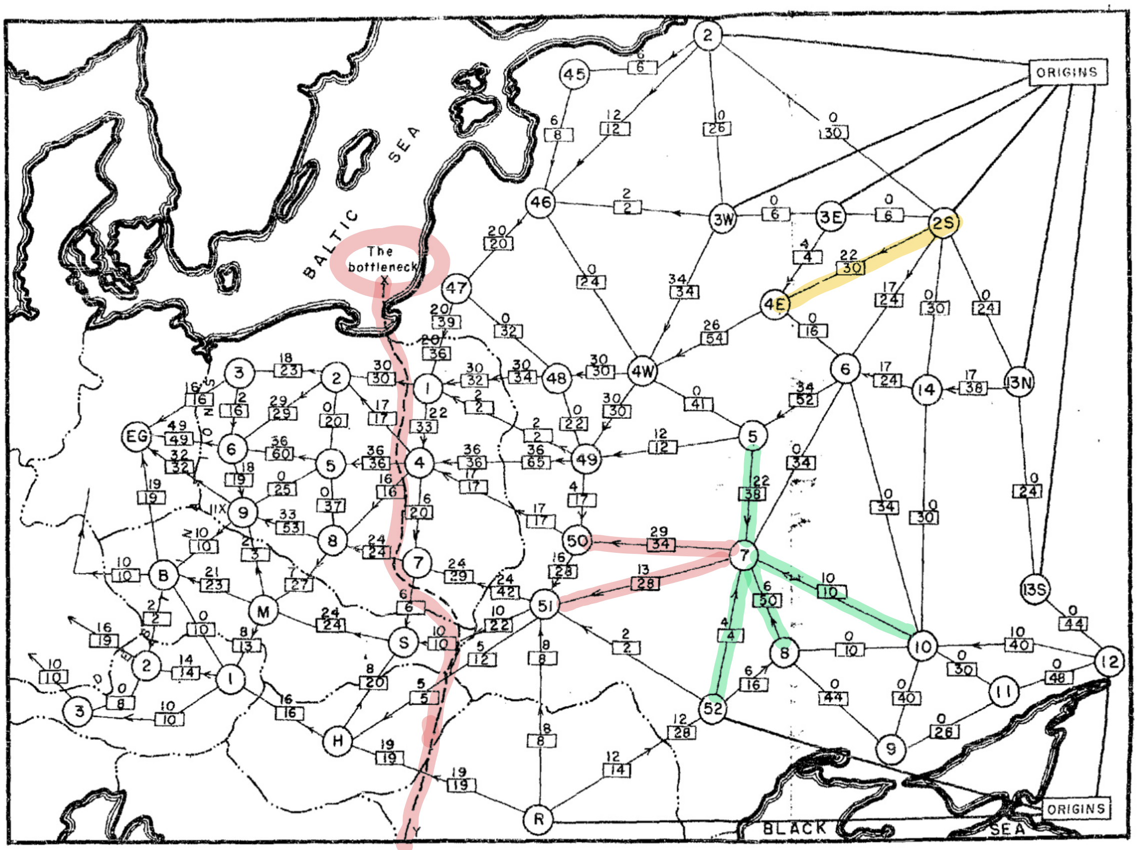

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Fig. 7 — Traffic pattern: entire network available

Legend:

International boundary

(8) Railway operating division

Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in trains / 1000's of tons } each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovakia); and 2, 3 (Austria)

Alternative destinations: Germany or East Germany

Note IIX at Division 9, Poland

Maximum flow: Given $G=(V,E)$ directed

capacity function $c: E \to \mathbb{R}_{\geq 0}$

two vertices source s & target t

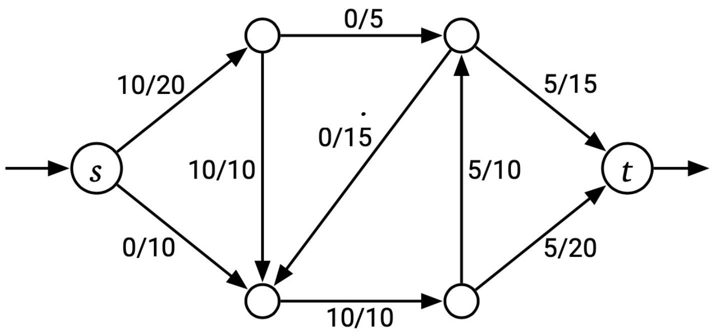
We want: <u>Flow</u> function $f: E \to \mathbb{R}$

Conservation s.t. $\sum_u f(u \to v) = \sum_w f(v \to w)$ for all $v \neq s, t$

feasible s.t. $0 \leq f(e) \leq c(e)$

maximize $|f| = \sum_w f(s \to w) - \sum_u f(u \to s)$

$= \sum_u f(u \to t) - \sum_w f(t \to w)$

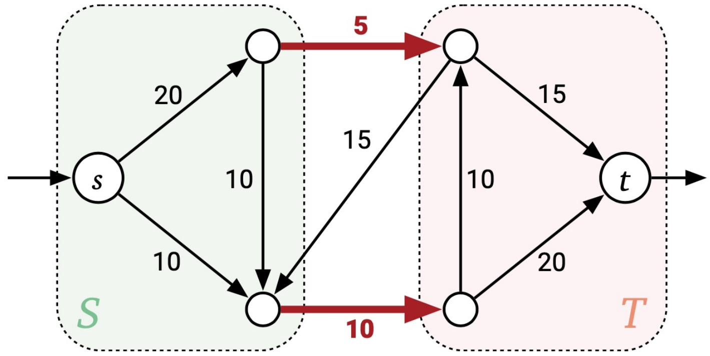


Flow/capacity

Minimum cut: Same input

cut Partition $V = S \sqcup T$

capacity $\|S,T\| = \sum_{\substack{u \in S \\ v \in T}} c(u \to v)$ as small as possible



MaxFlow-Mincut Theorem

$$\max |f| = \min \|S,T\|$$

First $\leq$

Pick any feasible flow f and any cut (S,T)

$$|f| = \sum_w f(s \to w) - \sum_u f(u \to s) \qquad \text{def } |f|$$

$$= \sum_{v \in S} \sum_{w \in T} f(v \to w) - \sum_{v \in S} \sum_{u \in T} f(u \to v) \qquad \text{conservation}$$

$$\leq \sum_{v \in S} \sum_{w \in T} f(v \to w) \qquad \text{because } f \geq 0$$

$$\leq \sum_{v \in S} \sum_{w \in T} c(v \to w) \qquad \text{because } f \leq c$$

$$= \|S,T\| \qquad \text{def } \|S,T\|$$

$\Rightarrow$ If $|f| = \|S,T\|$ then

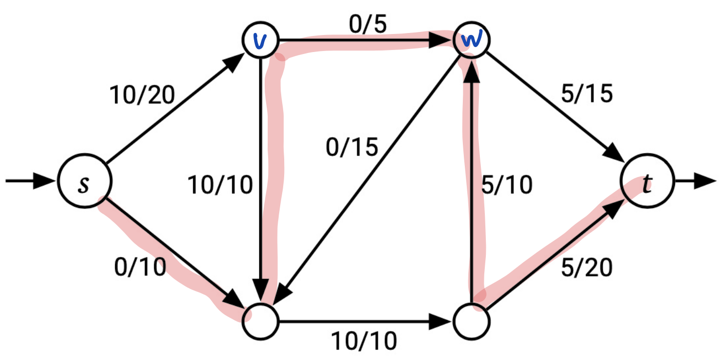
- f is a max flow
- (S,T) is a min cut

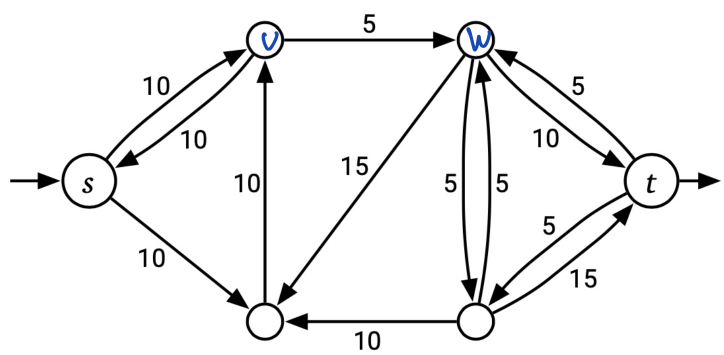
- $f(u \to v) = c(u \to v)$ for all $u \in S$ $v \in T$
- $f(u \to v) = 0$ for all $u \in T$ $v \in S$

Proof of $\geq$

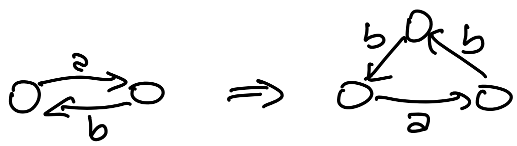
Pick your favorite flow f

Define
$$c_f(u\to v) = \begin{cases} c(u\to v) - f(u\to v) & \text{if } u\to v \in E \\ f(v\to u) & \text{if } v\to u \in E \end{cases}$$



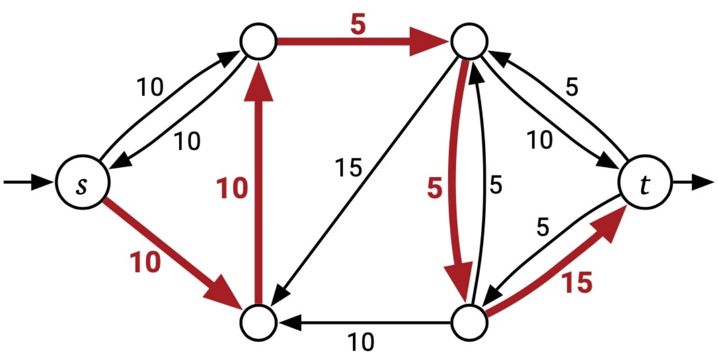


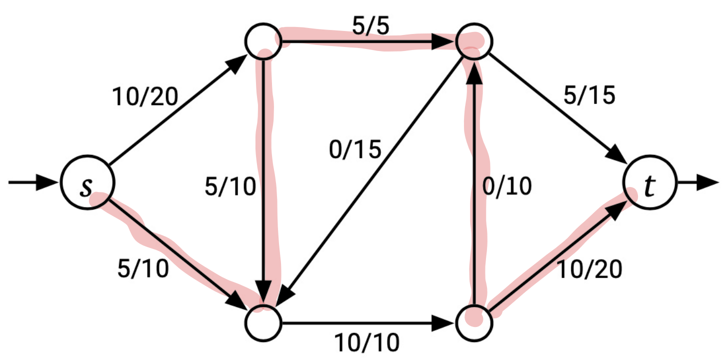
residual graph G_f



Case 1: s can reach t in G_f

c_{min} = min capacity of edges along any path from s to t





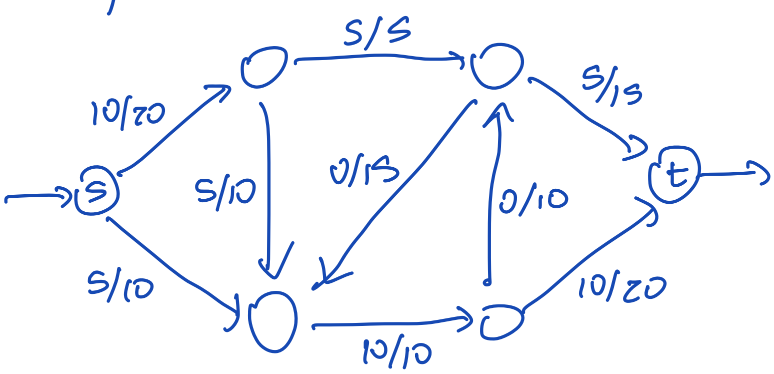
Push c_{min} units of flow along path

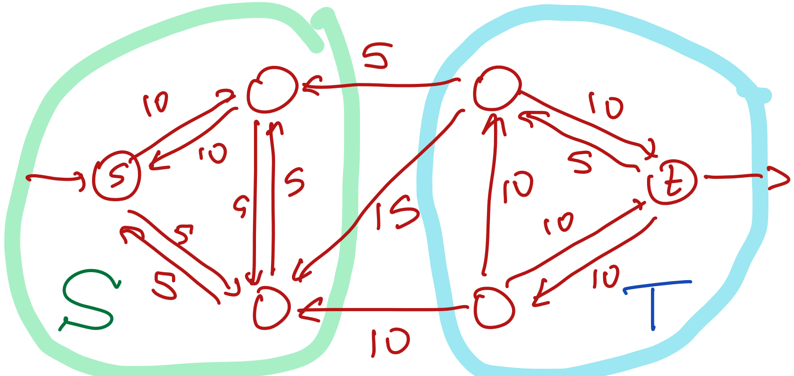
we get new flow f' s.t. $|f'| = |f| + c_{min}$

f' is feasible

$\Rightarrow$ f is not a max flow!

Case 2: No path from s to t in G_f





Let S = all vertices reachable from s in G_f

$T = V \setminus S$

For every vertex $u \in S$ $\quad v \in T$

if $u \to v \in E$ $\quad f(u \to v) = c(u \to v)$

if $v \to u \in E$ $\quad f(v \to u) = 0$

$\Rightarrow$ f is a max flow

and (S, T) is a min cut!

Ford Fulkerson '52

Augmenting path algorithm

```
f ← 0
G_f ← G
while there is a path P from s to t in G_f
    push flow along P
    rebuild G_f
return f
```