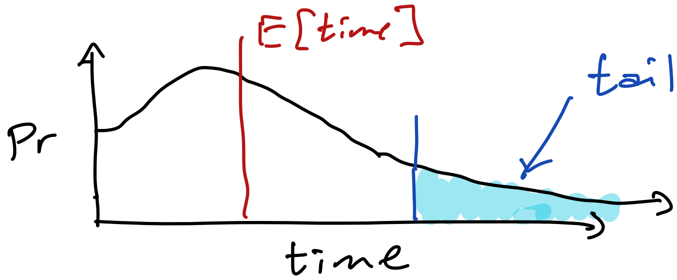
Quicksort runs in $O(n \log n)$ ~~expected~~ time with high prob.

Treap search runs in $O(\log n)$ ~~expected~~ time with high prob.



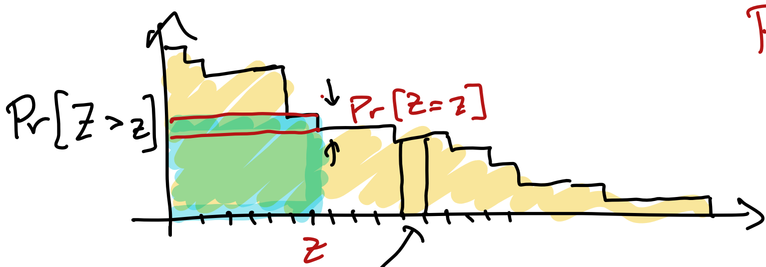
tail inequalities

$$X = X_1 + X_2 + \cdots$$

$$\Pr[X \geq \alpha \cdot E[X]] < \_\_\_\_\_\_$$

Markov's Inequality: If Z is any non-negative integer r.v.

$$\Pr[Z > z] \leq \frac{E[Z]}{z} = \frac{\mu}{z}$$



$$\sum_z \Pr[Z > z] = \sum_z z \cdot \Pr[Z = z] \geq z \cdot \Pr[Z > z]$$

$$\geq \sum_{i=0}^{z-1} \Pr[Z > i] \geq \sum_{i=1}^{z} \Pr[Z > z] =$$

$$\Pr[X > \alpha E[X]] \leq \frac{1}{\alpha}$$

$$\Pr[\text{Quicksort runs} > n^3 \text{ time}] \leq \frac{n \log n}{n^3} \sim \frac{1}{n^2}$$

X and Y are independent iff

$$\Pr[X = x \text{ and } Y = y] = \Pr[X = x] \cdot \Pr[Y = y]$$

$$\Pr[X = x \mid Y = y] = \Pr[X = x]$$

$$\Longrightarrow \boxed{E[X \cdot Y] = E[X] \cdot E[Y]}$$

$\Longrightarrow$ $f(X)$ and $f(Y)$ are independent

$X_1, X_2, \ldots X_n$ are fully independent

$$\Pr\left[\bigwedge_{i=1}^{n} X_i = x_i\right] = \prod_{i=1}^{n} \Pr[X_i = x_i]$$

$X_1, X_2, \ldots X_n$ are k-wise independent if every subset of size k is fully indep.

Example X_1, X_2 indep fair coins
$X_3 = X_1 \oplus X_2$

X_1, X_2, X_3 are pairwise indep (but <u>not</u> fully)

$$X = \sum_{i=1}^{n} X_i \qquad X_i \in \{0,1\}$$

$$p_i = \Pr[X_i = 1] = E[X_i]$$

$$\mu = E[X] = \sum_{i=1}^{n} p_i$$

<u>Chebyshev's inequality:</u>

If $X_1 \ldots X_n$ are pairwise indep, $\Pr[\underbrace{(X-\mu)^2}_{\text{2nd moment}} \geq z] \leq \frac{\mu}{z}$

<u>Proof:</u>
Let $Y_i = X_i - p_i \qquad Y = \sum Y_i = X - \mu$

$$E[Y^2] = E\Big[\sum_{i,j} Y_i Y_j\Big] = \sum_{i,j} E[Y_i Y_j]$$

$$= \sum_i E[Y_i^2] + \sum_{i \neq j} E[Y_i Y_j]$$

$$= \sum_i E[Y_i^2] + \sum_{i \neq j} \underbrace{\cancel{E[Y_i]}}_{0} \cdot \underbrace{\cancel{E[Y_j]}}_{0}$$

$$= \sum_i \left(p_i(1-p_i)^2 + (1-p_i)(-p_i)^2\right) \sim \leq \mu$$

Markov: $\Pr[Y^2 \geq z] \leq \frac{E[Y^2]}{z} \leq \frac{\mu}{z}$ $\square$

$$\Pr[X \geq (1+\delta)\mu] \leq \frac{1}{\delta^2 \mu}$$

$$\Pr[X \leq (1-\delta)\mu] \leq 1/\delta^2\mu$$

<u>Exponential Moment Inequality:</u>

If $X_1 \ldots X_n$ are fully independent

then $E[\alpha^X] \leq e^{(\alpha-1)\mu}$ for any $\alpha > 1$.

$$\Pr[X \geq x] \leq e^{x-\mu}\left(\frac{\mu}{x}\right)^x \qquad \text{Proof: Set } \alpha = \frac{x}{\mu}.$$

$$\Pr\left[X \geq (1+\delta)\mu\right] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu} \leq e^{-\delta^2\mu/3}$$

Treaps: $E[\text{depth}(k)] = \sum_{i=1}^{n} \Pr[i \Uparrow k]$

$X = \text{depth}(k)$ $\qquad X_i = [i \Uparrow k]$

Claim: $[1 \Uparrow k]\,[2 \Uparrow k] \cdots [k\text{-}1 \Uparrow k]$ are fully independent

$$\Pr[\text{depth}(k) > 8 \ln n] \leq \frac{2}{n^{2.5\text{ish}}}$$

$$\Pr[\max_k \text{depth}(k) > 8 \ln n]$$
$$< \sum_{k=1}^{n} \Pr[\text{depth}(k) > 8 \ln n] \leq \frac{2}{n^{1.5\text{ish}}}$$

$\Downarrow$

$$E[\max \text{depth}] = O(\log n)$$

Problem 3:

0 0 1 [0 1 0 1 1 1 0] 0 0 0 1 1

0 1 0 1 0 0 1

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For any shift value s:

$$HD(s) = \sum_{i=1}^{m} \left[P[i] \neq T[i+s] \right]$$

$(P[i] - T[i+s])^2$

$$= \sum_{i=1}^{m} \underbrace{P[i]}_{a_i} \underbrace{(1 - T[i+s])}_{b_{i+s}} + \sum_{i=1}^{m} (1 - P[i]) T[i+s]$$

$a = a_0\, a_1 \cdots a_m$

$b = b_0\, b_1 \cdots b_n$

$(a * b) = c = c_0 \cdots c_{n+m}$

$$c_j = \sum_i a_i \cdot b_{j-i}$$

$a_i = P[m-i]$

$b_j = 1 - T[j]$

$$\sum_{i=1}^{m} P[i] \cdot (1 - T[i+s]) = \sum_i a_{m-i} \cdot b_{s+i} = (a * b)_{s+m}$$